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The AMERICAN MATHEMATICAL MONTHLY

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Ralph D. James, *Editor*

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PASCAL AND THE INVENTION OF PROBABILITY THEORY

OYSTEIN ORE, Yale University*

Most textbooks on probability feel obliged to include a brief account of the history of the theory. Their descriptions of this process of initiation usually run somewhat in the following vein: "In the year 1654 a gambler named de Méré proposed to Pascal two problems which he had run across in his experiences at the gaming table."

It is likely that the distinguished Antoine Gombaud (or Gombauld), chevalier de Méré, sieur de Baussay, would turn in his grave at such a characterization of his main occupation in life. He certainly considered himself a model of courtly behavior and taught his esthetic principles elegantly to the *haut monde* as one may see from the frontispiece of his collected works. His writings appear today a little humorless and pedantic, but parts are still sufficiently entertaining to be readable, and they have secured him a permanent niche in the French literature of the seventeenth century.

De Méré (1607–1684) had received a good classical education and had served briefly in the army. His time was divided about equally between his small estate in Poitou and the court at Paris. His works show him to be a philosopher who in popular form expounded the ethics of a noble life, with particular emphasis upon the *agrêments* and the pleasant considerations for others which are essential for the *honnêtes hommes* of high society. He rapidly became a prominent figure at the court of Louis XIV where he was an adviser in delicate situations and an arbiter in conflicts. His charm, good taste, art in conversation, and correspondence made him an attractive guest in the salons and a friend of many of the important figures of his period. He recalled with particular pride his assistance to Madame de Maintenon before she became the favorite of the king. As time wore on he seems occasionally to have had rather exaggerated ideas of his own importance.

Sociability was his ideal; the criterion for good conversation was that it should be pleasant. Specialists were an abomination to him: "Most of them do not instruct at all since they rely upon obscure, sometimes even false, principles, and instead of seeking the truth to clarify it they aim to embarrass each other, also by terms which they do not even themselves understand, and by chimerical distinctions." It may even seem a little *mal à propos* for a mathematician to lecture on him, in view of de Méré's contempt for the scholars. He describes some categories of them as follows: "The best mathematicians who do not know how to entertain us except by numbers and figures, those knowing history by heart without ever having reflected upon it, or those who have a curious knowledge of many languages without having anything to say in them."

There is a little of this disdain in de Méré's description, in one of his letters, of how he made the acquaintance of Pascal: "I once made a trip with the duke

* The present article is based upon a lecture given on February 25, 1959, at Colorado College on the occasion of the centennial of the birth of Florian Cajori (Feb. 28, 1859–Aug. 14, 1930).

of Roannez, who used to express himself with good and just sense and whom I found good company. Monsieur Mitton, whom you know and who is liked by all at court, was also with us, and because the trip was supposed to be a promenade rather than a voyage, we only thought of enjoying ourselves and we discussed everything. The duke was interested in mathematics, and in order to relieve tedium on the way he had provided a middle-aged man, who then was very little known, but who later certainly has made people talk about him. He was a great mathematician who knew nothing but that. These sciences give little sociable pleasure, and this man, who had neither taste nor sentiment, could not refrain from mingling into all we said, but he almost always surprised us and often made us laugh." De Méré goes on to tell that Pascal also carried strips of paper which he brought forth from time to time to write down some observations. After a few days Pascal came to enjoy the company and talked no more of mathematics.

This trip to Poitou probably took place in 1651 or 1652, but de Méré's account was written many years later and it seems that, in reminiscing, the chevalier's memory must have failed him to some extent. At the time, Blaise Pascal (1623–1662) was not yet thirty years old and could hardly be called a middle-aged man. Of course, he was almost constantly ill and may have aged prematurely. But he was already a well-known scientist. As a child prodigy he had accompanied his father to the meetings of the *Académie libre* in Paris, and when sixteen years old he had published a remarkable treatise on conic sections. At eighteen he had caused a great stir through the invention of his calculating machine, this *machine arithmétique* by means of which " . . . one could not only do all sorts of reckoning without feather or casters, but they could be done infallibly even if one did not know any of the rules of arithmetic." Pascal's demonstration of the weight of the atmosphere by barometric measurements at the base and summit of the Puy de Dôme peak had placed him in the forefront among contemporary physicists. Yet, in spite of all these achievements, Pascal's renown in Paris was still small in comparison with the fame he was later to acquire as the author of the *Lettres provinciales*. Perhaps, in retrospect, this was what the chevalier had in mind.

Pascal was at this time at the beginning of his so-called "worldly period," and, in his further account, de Méré even takes some credit for having brought it on. Certainly the two became well acquainted, and Pascal was influenced by de Méré's ideas on literary style. It has been indicated that de Méré advised Pascal on strategy in his attacks on the Jesuits in some of the *Lettres provinciales*.

The extent to which Pascal participated in the pleasures of life in the *haut monde* has been much argued by his biographers. His pious sisters feared that he was on the path to perdition. His interest in gaming and gambling questions have been cited as evidence of his dissipation in this period. Probably both Pascal and his friend de Méré spent some time at play; it was the fashionable pastime. But there is no indication that they did so with any passion; on the

contrary, in their writings they both express themselves rather contemptuously against gambling. De Méré wrote to their common acquaintance Mitton about the pleasures of the countryside—two months is all he can stay in Paris before he gets homesick. "But I confess also that on my part I deplore you who are confined to gambling, longing for nothing but luck, without eyes for anything but this artificial world, almost like the courtesans to whom the great beauties of nature are unknown."

Pascal on his side tries to analyze the gambler's soul in the *Pensées*. "But you may say, what is his object? To boast tomorrow to his friends that he has played better than another? Such a man relieves the tedium of his life by playing every day. If you were to give him in the morning the money he might win during the day on condition that he should not gamble, you would make him unhappy. One may think possibly that he seeks the entertainment only of the game and not the gain. But let him play for nothing and he is bored and does not warm up to it. Therefore, it is not only the amusement which he seeks—languishing play without passion he finds tedious. He must become excited and deceive himself into believing that he would be happy to win that which he would not even accept were it not for the play. He must form an object for his passion to excite his desire, his anger, his fear, just like children who are scared of their own faces when they have blackened them."

Next let us turn to the two probability problems for which Pascal actually found solutions. It has often been stated that they were based upon de Méré's personal gambling experiences. This, as we shall see, seems very unlikely. While he was the first to call Pascal's attention to them, they were old and well-known questions. We shall take up first the so-called *dice problem*: When one throws with two dice, how many throws must one be allowed in order to have a better than even chance of getting two sixes at least once? Games of this kind were evidently popular in the middle ages; Cardano had dealt with them more than a century earlier, and there are still French dice games of a similar nature.

Let us recall briefly how the usual solution is obtained. It is convenient to determine first the probability of not obtaining any sixes. If one throws once there are thirty-six different possible throws with two dice and thirty-five of these do not give two sixes. Thus the probability of not getting two sixes in one throw is $q_1 = 35/36$. If one throws twice, there are 36×36 cases and 35×35 of them do not give two sixes either time. Thus $q_2 = (35/36)^2$. In the same way one finds that in n throws the probability of not getting any two sixes is $q_n = (35/36)^n$. Hence the opposite event, that of getting two sixes at least once, has the probability

$$p_n = 1 - q_n = 1 - \left(\frac{35}{36}\right)^n.$$

To have a better than even chance one must have $p_n > \frac{1}{2}$ and one finds

$$p_{24} = .4914, \qquad p_{25} = .5055.$$

Thus, only if one has 25 or more throws is it an advantageous proposition.

De Méré believed that the smallest advantageous number of throws should be 24. As the matter has been presented, he turned to Pascal because his own experiences had shown him that 25 throws were required. This is an unreasonable explanation. The difference between the probabilities for 24 and 25 throws is so small, as we have just seen, that to decide experimentally that one of them is less than $\frac{1}{2}$ would, according to modern statistical standards, require at least 100 sequences of trials, which in turn would involve several thousand individual throws with the two dice. Besides, the dice would have to be specially made in order to show no bias; the usual bone cubes turned out by the diciers of Paris would be much too inaccurate. To prepare special equipment of this kind and to keep the tedious records involved was evidently contrary to the chevalier's temperament.

However, Pascal's letters on probability to Pierre de Fermat (1601–1665), the learned jurist in Toulouse, throw light on the subject. In reply to an earlier letter from Fermat, Pascal writes, on July 29, 1654, "I admire much more your method for the division problem than that for the dice problem. I know that several persons have found the solution of the dice problem, as for instance, Monsieur le chevalier de Méré, who was the one who proposed these questions to me, and also M. de Roberval. But M. de Méré has never been able to determine the correct value in the division problem, nor the method to solve it, so that I found myself to be the only one who knew this proposition."

A little later in the same letter Pascal reports further on de Méré's views:

> "He told me that the figures were wrong for the following reason: If one wants to throw a six with one die one has an advantage in four throws, as the odds are 671 to 625. If one shall throw two sixes with two dice there is a disadvantage in having only 24 throws. However, 24 to 36 (the number of cases for two dice) is as four to six (the number of cases on one die).
>
> This was a great scandal which made him proclaim loudly that the theorems were not constant and Arithmetic belied herself. But you can easily see the reason for this result by the principles you possess."

Pascal does not understand de Méré's reasoning, and the passage also has been unintelligible to the biographers of Pascal. However, de Méré bases his objection upon an ancient gambling rule which Cardano also made use of: One wants to determine the *critical number* of throws, that is, the number of throws required to have an even chance for at least one success. If in one case there is one chance out of N_0 in a single trial, and in another one chance out of N_1, then the ratio of the corresponding critical numbers is as $N_0 : N_1$. That is, we have

$$n_0 : N_0 = n_1 : N_1.$$

This immediately gives the proportion stated by de Méré. The rule was first proved by Abraham de Moivre (1667–1754) in his *Doctrine of Chances* (1716). If the chances are one in N_0 in a single trial, then the critical number is, with good approximation when N_0 is not too small,

$$n_0 = (\text{not } \log 2) \times N_0 = .69 \times N_0.$$

De Moivre applied this to the so-called Royal Oaks lottery in London; here there was one chance in 32, and so the critical number of trials by this rule is found to be $n_0 = 22.08$. The actual value is $n_0 = 22.135$, and, in general, de Moivre's rule gives very good values when N_0 is fairly large. However, de Méré made the error of believing that his gambling proportion was an absolute rule; for a small value, like $N_0 = 6$, the approximation to the actual value is not good enough.

These observations all indicate strongly that de Méré did not turn to Pascal with an actual gambling experience; rather, he was confused by the fact that a seemingly well-established gambling rule did not conform with the theoretical calculations which had been made. This tends to confirm the hypothesis that probability theory was at this time not in the state of absolute nonexistence that one is often led to believe. Cardano, around 1525, had already discovered certain rules which made it possible to solve the dice problem exactly, for one die. More than fifty years before Pascal, Galileo had given a complete table of probabilities for all throws with three dice. It appears likely that also in Pascal's circle in the mathematical academy, the simplest probability considerations were known. Pascal states that in addition to Fermat and himself, also de Méré and the mathematician Roberval could solve the dice problem.

In the preserved letters in the correspondence with Fermat, Pascal never refers to his own solution of the dice problem. Nor is it mentioned in Pascal's treatise on the *Arithmetic Triangle*, which was composed at this time and which includes the solution of the division problem as well as a few general probability principles. In regard to all other scientific achievements, Pascal always appeared anxious to receive proper recognition; in his pleasant correspondence with Fermat he insists on the importance of his own method in the division problem. In his investigations on the vacuum, he engaged in considerable argument to establish his priority, and the same is true in his later dispute regarding the cycloid or roulette curve. Thus there seems to be reason to believe that had Pascal had any feeling that this was an important discovery he would have expressed himself quite explicitly in regard to his priority rights.

The second problem which de Méré proposed was the *problème des parties*, commonly called the *division problem*. The question is how one shall divide equitably the prize money in a tournament in case the series for some reason is interrupted before it is completed. As we would say now, it is a question of determining the probability to win for each contestant, at a stage where each has a certain number of games or points to go. This is a problem of such difficulty that its solution by Pascal may well be considered a decisive break-through in the history of probability theory.

In this case it is still more evident that de Méré was not proposing a question from his own experience. For three centuries it had been a standard problem in mathematical texts. The first printed version may perhaps be found in Fra

Luca Paciuoli's *Summa* (1494), where, among the amusement questions, he proposes the following. "A team plays ball such that a total of 60 points is required to win the game, and each inning counts 10 points. The stakes are 10 ducats. By some incident they cannot finish the game and one side has 50 points and the other 20. One wants to know what share of the prize money belongs to each side. In this case I have found that opinions differ from one to another, but all seem to me insufficient in their arguments, but I shall state the truth and give the correct way."

His second example is a shooting match. "Three compete with the cross bow and the one who first obtains six first places wins; they stake 10 ducats among themselves. When the first has four best hits, the second three, and the third two, they do not want to continue and decide to divide the prize fairly. One asks what the share of each should be." Fra Luca also warns against gamblers who play "morra" to five points for five ducats and when they are behind, for instance four wins to three, " . . . they say, we come back . . . ," and want the prize divided three to two. Otherwise there is a noticeable avoidance of formulating the problem for a straight gambling game, presumably to avoid criticism for dealing with objectionable pastimes.

Most accounts of the division problem take their starting point in Fra Luca's examples. However, the problem is a much older one. I have found it in Italian mathematical manuscripts as early as 1380. It seems likely that it is of Arabic origin. It does not appear in Leonardo Fibonacci's *Liber abaci* (1202), which brought many Arabic puzzles to Italy, but the form of the problem is reminiscent of the distribution and inheritance problems of the Arabs.

The Renaissance mathematicians made only trivial contributions to the division problem, although those who deal with it make great claims for their own methods and are liberal in their criticisms of others. Cardano, for instance, says about the solution by Paciuoli, "And there is an evident error in the determination of the shares in the game problem as even a child should recognize, while he (Paciuoli) criticizes others and praises his own excellent opinion."

Cardano's arch enemy, Tartaglia, feels himself on swaying ground when he deals with the division problem in his *General Trattato* (1556). The margin displays the warning, "*Error di Fra Luca dal Borgo*," and Tartaglia gives his own rule, but with the reservation: "Therefore I say that the resolution of such a question is judicial rather than mathematical, so that in whatever way the division is made there will be cause for litigation." Toward the end of the chapter, Tartaglia considers these matters as having "*poco sugo*" and giving rise "to great dispute and so it appears to me better not to speak more of this matter, although some people like such facetious questions in order to have an occasion to create an argument."

The division problem remained in the arithmetic texts until well into the seventeenth century. It can be found also in French books, and de Méré may have read it in his own school. The form changed somewhat with time; let me cite a couple of examples from an arithmetic by Forestani (Venice 1603). "An

elderly nobleman, staying at his country house, was extremely fond of watching ball games, and so he called in two young farmhands, saying, 'Here are four ducats for which you may play; the one who first takes eight games is the winner.' So they began to play, but when one had five games and the other three games, they lost the ball and were unable to finish. The question is how the prize should be divided."

Another runs as follows: "Three soldiers garrisoned at a fortress take a walk and find a scudo. Each of them claims it; however at last they agree that they will play a match of pallatella to fourteen games, and he who wins shall have the scudo." When they have won respectively ten, eight, and five games, they are called to guard duty and the proper shares shall again be determined.

Pascal probably found the solution of the division problem early in the spring of the year 1654. The method depends on binomial coefficients, and Pascal was undoubtedly greatly aided by his studies of the *Arithmetical Triangle*, a table of such coefficients. The solution was based upon a somewhat artificial approach. When the two players needed a and b games respectively to win, Pascal lets them play altogether $a+b-1$ games, regardless of whether the series might have been decided before this many games. The argument is quite correct and is still in common use in textbooks on probability. However, the mathematician Roberval objected strenuously, and a discussion with Roberval was usually not pleasant. One of his contemporaries called him "the greatest mathematician in Paris, and in conversation the most disagreeable man in the world."

These criticisms by Roberval seem to have been the immediate reason why Pascal sought the opinion of Fermat, the recognized grand master of mathematics in France at the time. Carcavy, the royal librarian, also a member of the scientific circle in Paris, acted as an intermediary; he had formerly been Fermat's colleague as a judge at the parliament court in Toulouse.

It will carry us too far to give a detailed account of this correspondence, which lasted through the summer and fall of 1654. Fermat was delighted to come into closer contact with the young Pascal; he had previously been on friendly terms with his father. Fermat had begun to feel the scientific isolation in Toulouse and hoped that Pascal might assist him in publishing his mathematical results. In one of his letters to Carcavy he writes, "I have been delighted to have my own sentiments conform to those of M. Pascal, for I admire his genius infinitely and believe he is capable of achieving anything he may undertake. The friendship which he offers to me is so precious and so considerable that I believe it will not be disturbed if I should make some use of it in the printing of my treatises."

Pascal, on the other hand, was cheered by finding that Fermat's results were in complete agreement with his own: "I see that the truth is the same in Toulouse and Paris." Pascal at this time was absorbed in an intense scientific production; among other things he completed his *Traité du Triangle Arithmétique* with an extensive discussion of the division problem. After his death

this work was found fully printed, but unpublished, among his posthumous papers. The reason is well known. On November 23, 1654, in Paris, the crisis occurred which has been called Pascal's definitive conversion. From now on his life and thoughts revolved almost exclusively around the questions of his Christian faith. He joined his sister as a member of the Jansenist group for whom scientific studies were but illusions serving to distract from the final purpose of life, the salvation of the soul.

But Pascal never quite relinquished his interest in the newly created field of probability. In his famous *Pensées*, one of the most curious sections is "*Le pari*," the wager, a dialogue about the existence of God. It is at first difficult to understand and it has been widely discussed. But if one recognizes that Pascal has a definite mathematical probability formula in mind, the passage becomes quite lucid. It has been conjectured that Pascal conceived of "*Le pari*" as a dialogue between himself and his unbelieving friend de Méré:

> Pascal: God exists or he does not. Which side shall we take. Reason can decide nothing. An infinite chaos separates us. A game is being played where a decision, heads or tails, will be made at the end of this infinite distance. On what do you place your bet? By reason you cannot take one or the other; by reason you can defend neither choice. Therefore, do not blame the error of those who have made a choice, since you know nothing about it.
>
> De Méré: No, but I blame them for having made a choice at all, not for their particular choice, for they are equally at fault, both he who chooses heads and he who chooses tails. The correct attitude is not to bet at all.
>
> Pascal: Yes, but one is compelled to wager, it is not voluntary, you are in the game. Which side do you take? Let us see. Since you must wager, let us find out which alternative is the least profitable.

Pascal goes on to argue, on the principle of mathematical expectation, that the value of a game is the prize to be won times the probability for winning it. This should be compared to the amount risked times the probability for losing.

> Let us see: since there is an equal chance of gain or loss, and if you were to win only two lives for one, you should still bet. But if there were three to win you would also play—since you are compelled to—and you would be imprudent not to risk your life to win three others in a game with such chances to win or lose. But there is an eternity of life and happiness. And when this is so, if there should be an infinite number of chances with only a single favorable to you, it would still be right to bet one to obtain two; you would act with bad judgment if, when obliged to play, you would refuse to stake one life against three, even if there is an infinity of chances and but one for you, provided there is an infinite life of infinite happiness to be gained. But here, actually, there is such an infinite life of infinite happiness to be won, one chance of winning against a finite number of possibilities for a loss, and that which you risk is finite. This eliminates all choice; whenever an infinite gain is involved and there is not an infinite number of losing chances against the winning ones, there is nothing to weigh, one must give all. And so, when forced to play, one must sacrifice reason to win life rather than to stake it against the infinite profit which may accrue just as easily as the loss: annihilation.

Pascal continues in the same vein, and in the end de Méré seems to be convinced, but finally he asks somewhat irreverently, "I confess, this I admit, but then is there no way of looking at the underside of the cards which have been dealt?" To this Pascal replies, "Yes, the Holy Scriptures and the rest."

Much criticism has been expressed against Pascal's logic, and one must admit that this is hardly a field for applied mathematics. But even Pascal himself cannot have believed that his argument should carry a convincing mathematical weight; however, if his design was to create a thought-provoking parable, he succeeded admirably.

It was well known in Paris that Pascal and Fermat had disccovered a new branch of mathematics, but no one knew much about the details. The young Dutch genius Christiaan Huygens arrived in Paris less than a year later. He had worked on a couple of problems concerning games of chance and was anxious to consult with one of the principals in probability theory. Huygens was well received in Paris. He was introduced to the members of the informal academy, and made the acquaintance of Roberval and the lawyer and amateur mathematician Mylon. But Fermat was far away in Toulouse, and after his conversion Pascal admitted no visitors. Huygens did not receive the information he desired, but after his return to Holland he began drafting his own little treatise on probability, *Calculations in Games of Chance*. But he was concerned about the correctness of his own results, and to obtain a check on them he sent one of his problems to Roberval, Mylon, and Carcavy. Mylon replied with an erroneous answer which Huygens politely corrected, and from Roberval we have no report. Carcavy, however, consulted with his close friend Pascal, and also forwarded the problem to Fermat. Fermat promptly confirmed Huygens' solution and included for Huygens' consideration a series of five problems, which are reproduced at the end of the little treatise.

Encouraged by this contact, Huygens wrote another letter to Carcavy and a little later, much to his surprise, he received a letter from Mylon stating that Pascal had found his principle admirable and in conformity with his own procedure. Mylon explained that "although it is very difficult to meet Pascal since he has retired completely to give himself entirely to devotion, he has not lost his mathematics from view. When M. de Carcavy can visit him and propose some problem to him, he does not refuse to give the solution, particularly in the field of games of chance which he was the first to bring under discussion. Since I do not possess the same goodness as these gentlemen, I have all the difficulties in the world to meet them, since they are entirely absorbed in religious affairs and I only rarely visit those places."

Equally surprising was the fact that Pascal had given Carcavy a gambling problem to transmit. "*A* and *B* play at hazard with three dice and fixed points fourteen and eleven respectively. Each has twelve pennies and receives one penny from the other every time his own point turns up. What are the odds for one player to ruin the other?"

This problem Huygens included as the last in his collection of exercises for the readers of his booklet on probability. It is far more difficult than the rest, and it embodies, in spite of its innocuous form, the beginnings of a whole field of probability, the theory of random walks, Brownian motion, and other questions from the kinetic gas theory.

Pascal's moral views evidently did not permit him to propose a plain gambling proposition, so that in his original version as reported by Carcavy he only lets the players make pencil marks on paper. Pascal was keenly aware of the difficulties involved and he had the hope that he had invented a problem which might stymie even the formidable Fermat. In the correspondence between the two, one has a feeling that Pascal is a little dismayed at the apparent ease with which the old master tackled his problems. But also this time Fermat immediately returned a solution which agreed with the one found by Pascal.

Huygens was elated at the confirmation of his methods, but regretted deeply his failure to meet Pascal: "If one had not assured me while I was in Paris that he had abandoned the study of mathematics entirely I should have tried by every means to make his acquaintance." Later Huygens corresponded with Pascal on other matters, and when Huygens returned to Paris in 1660, a couple of years before Pascal's death, the two met on several occasions.

Huygens also became friendly with the duke of Roannez and in his diary he relates that he was once entertained at dinner in the ducal palace in the company of the chevalier de Méré, "inventor of the division in games." It is noteworthy that in all the discussion about probability problems no one thought it worth while to consult with de Méré. Pascal, in one of his letters to Fermat, made the comment upon him, "He is a good wit, but not a mathematician."

Nevertheless, the fact that de Méré had been a figure, albeit a minor one, in the creation of a new mathematical field seems to have gone to his head. To the consternation of contemporary scientists he wrote a letter to Pascal in the following vein.

"Do you remember you once told me that you were no longer convinced of the excellence of mathematics? You write to me this time that I have disillusioned you completely and also that I have discovered things which you would never have perceived if you had not known me. I don't know, however, Monsieur, if you are as obliged to me as you may think. You still have the habit, which you have gathered from this science, not to judge anything except from your demonstrations, which are often false. These long reasonings drawn from line to line prevent you from obtaining the higher point of view which never deceives."

Later on he admonishes Pascal as follows:

"You know that I have discovered such rare things in mathematics that the most learned among the ancients have never discussed them and they have surprised the best mathematicians in Europe. You have written on my inventions, as well as Monsieur Huygens, Monsieur de Fermat, and many others who have admired them. You may conclude from this that I do not propose to anyone to scorn this science and truly, it may be of service provided one does not attach oneself too closely to it, for ordinarily, that which one seeks with so much curiosity appears useless to me and the time spent at it could be better employed."

De Méré's outburst seems to have amused the court and a wit proposed epigrammatically that the chevalier believed "he could teach Madame de Maintenon courtly behavior and Pascal mathematics."

Peculiarly enough, the whole letter was printed verbatim in Boyle's important encyclopedia *Dictionaire historique et critique*. When Leibniz came across

the article he commented:

"I almost laughed at the airs which the chevalier de Méré takes on in his letter to Pascal which Boyle reports in the same article. But I notice that the chevalier knew that Pascal's genius also had its weak sides which sometimes made him susceptible to the influence of too extravagant spiritualists and even at times made him lose the taste for solid knowledge.

M. de Méré takes advantage of this to talk down to Pascal. It seems to me that he makes a little fun of him, as men of the world often do when they have an abundance of esprit, but mediocre knowledge. They want to convince us that those things which they do not sufficiently understand are but of small value; one should send them to school with Roberval. It is true, nevertheless, that the chevalier was unusually gifted even in mathematics."

Leibniz goes on with a brief mention of some of the men who had worked with probability problems. He himself often showed an interest in the subject but never made any contributions of consequence. However, he concludes his epistle with the judgment which has never been more true than at present, "So also the games in themselves merit to be studied and if some penetrating mathematician meditated upon them he would find many important results, for man has never shown more ingenuity than in his plays."

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INTEGRATION BY PARTS FOR STIELTJES INTEGRALS

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There are several formulas for integration by parts. The most familiar of these, as found in the usual calculus text, states that

$$\int_a^b f(t)g'(t)dt + \int_a^b f'(t)g(t)dt = f(b)g(b) - f(a)g(a). \tag{1}$$

Clearly (1) does not hold without some restriction on the functions f and g. (We shall point out below the class of functions to which it is appropriate to apply (1).)

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For Riemann-Stieltjes integrals, there is a reasonable formula for integration by parts. Let f and g be finite real-valued functions defined on the closed interval $[a, b]$ of the real number system R†. Let $\int_a^b f(t)dg(t)$ denote the Riemann-Stieltjes integral of f with respect to g over $[a, b]$. Then, if $\int_a^b f(t)dg(t)$ exists, the integral $\int_a^b g(t)df(t)$ also exists, and

$$\int_a^b f(t)dg(t) + \int_a^b g(t)df(t) = f(b)g(b) - f(a)g(a). \tag{2}$$

For a proof of (2), see for example [1], p. 195. Formula (2) suffers from the inherent defect of Riemann-Stieltjes integrals: even for quite ordinary functions f and g, the integral $\int_a^b f(t)dg(t)$ may fail to exist (for example, if f and g have a common left or right discontinuity).

The appropriate vehicle for Stieltjes integration appears to be the Lebesgue-Stieltjes integral, discussed in detail in [4, Ch. 3] and in [3, Ch. IV]. For this integral there is a perfectly general formula for integration by parts. The formula for integration by parts is usually stated with certain restrictions on the integrand and integrator (see [4, p. 102, Theorem (14.1)] and [3, p. 161, Satz 1]). These restrictions are in fact unnecessary. We shall give what appears to us the proper general form of the formula for integration by parts.

We recall first a few notions from measure theory. Consider a closed interval $[a, b]$ on the line R, and the family $\mathfrak{F}$ of all closed subsets of $[a, b]$. There are families of subsets of $[a, b]$ that contain $\mathfrak{F}$ and also are closed under the formation of countably infinite unions and complements relative to $[a, b]$. (The family of *all* subsets of $[a, b]$ is such a family.) Let $\mathfrak{B}([a, b])$, the Borel sets in $[a, b]$, be defined as the *smallest* family of subsets of $[a, b]$ that contains $\mathfrak{F}$ and is closed under the formation of countable unions $\bigcup_{n=1}^{\infty} B_n$ and complements $[a, b] \cap B'$. In this note, we shall define a *measure* on $\mathfrak{B}([a, b])$ as a nonnegative, finite, real-valued function, say μ, defined for all sets in $\mathfrak{B}([a, b])$, such that

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu(A_n) \tag{3}$$

if $A_1, A_2, A_3, \cdots$ are sets in $\mathfrak{B}([a, b])$ and $A_n \cap A_m = \varnothing$ for $n \neq m$ ($\varnothing$ denotes the void set).

For every number t in $[a, b]$, the intervals $[a, t]$ and $[a, t[$ are obviously Borel sets, and hence $\mu([a, t])$ and $\mu([a, t[)$ are defined for all t, $a \leqq t \leqq b$, if μ is a measure on $\mathfrak{B}([a, b])$.

Now let μ and ν be any two measures on $\mathfrak{B}([a, b])$. For $a \leqq t \leqq b$, let us write

$$\begin{aligned} M(t) &= \tfrac{1}{2}\{\mu([a, t]) + \mu([a, t[)\}, \\ N(t) &= \tfrac{1}{2}\{\nu([a, t]) + \nu([a, t[)\}. \end{aligned} \tag{4}$$

† For real numbers a and b such that $a \leqq b$, we write $[a, b]$ to denote the closed interval $\{t: t \in R, a \leqq t \leqq b\}$, and $[a, b[$ to denote the half-closed interval $\{t: t \in R, a \leqq t < b\}$. The expressions $]-\infty, a]$ and $]-\infty, a[$ are defined similarly.

It is obvious that M and N are (weakly) monotone increasing, nonnegative functions defined on $[a, b]$. Since $\mu(\varnothing)=\nu(\varnothing)=0$, we see that $M(a)=\frac{1}{2}\mu(\{a\})$, $N(a)=\frac{1}{2}\nu(\{a\})$. We shall state the theorem on integration by parts in terms of the functions M and N.

THEOREM A. *Let μ and ν be measures on $\mathfrak{B}([a, b])$ and M and N the functions on $[a, b]$ defined in* (4). *Then**

$$\int_{[a,b]} M(t)d\nu(t) + \int_{[a,b]} N(t)d\mu(t) = \mu([a, b])\cdot\nu([a, b]). \tag{5}$$

Proof. Both integrals appearing in (5) exist. Since M and N are increasing real-valued functions on $[a, b]$, they are clearly Borel measurable and bounded; and bounded measurable functions are integrable. The proof of (5) is extremely simple. Let $[a, b]\times[a, b]$ denote the Cartesian product of $[a, b]$ with itself, that is, the closed square in the plane $\{(t, u): a\leqq t\leqq b,\ a\leqq u\leqq b\}$. Let Q be the set $\{(t, u): (t, u)\in[a, b]\times[a, b],\ t\geqq u\}$. One version of Fubini's theorem ([4, p. 85, Theorem (9.8)]) asserts that†

$$\int_{[a,b]} \mu\{t: t\in[a, b], (t, u)\in Q\}d\nu(u) = \int_{[a,b]} \nu\{u: u\in[a, b], (t, u)\in Q\}d\mu(t). \tag{6}$$

Equality (6) can be rewritten as

$$\int_{[a,b]} \mu([u, b])d\nu(u) = \int_{[a,b]} \nu([a, t])d\mu(t). \tag{7}$$

Since $[a, b]=[a, u[\cup[u, b]$ $(a\leqq u\leqq b)$, the left side of (7) can be rewritten as

$$\begin{aligned}\int_{[a,b]} \mu([u, b])d\nu(u) &= \int_{[a,b]} \mu([a, b])d\nu(u) - \int_{[a,b]} \mu([a, u[)d\nu(u)\\ &= \mu([a, b])\cdot\nu([a, b]) - \int_{[a,b]} \mu([a, u[)d\nu(u).\end{aligned} \tag{8}$$

Combining (7) and (8), we have

$$\int_{[a,b]} \mu([a, u[)d\nu(u) + \int_{[a,b]} \nu([a, t])d\mu(t) = \mu([a, b])\cdot\nu([a, b]). \tag{9}$$

Interchanging the rôles of μ and ν, we can also write

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* Here and below all integrals are of Lebesgue's type.

† Let $\phi=\phi(t, u)$ be the function defined on $[a, b]\times[a, b]$ such that ϕ is equal to 1 on the set Q and 0 off the set Q. Then (6) states that the iterated integrals $\int(\int\phi d\mu)d\nu$ and $\int(\int\phi d\nu)d\mu$ are equal. This very special case of Fubini's theorem can be proved by an elementary argument: see the Remark at the end of this paper.

$$\int_{[a,b]} \mu([a,u])d\nu(u) + \int_{[a,b]} \nu([a,t[)d\mu(t) = \mu([a,b])\cdot\nu([a,b]). \tag{9'}$$

Adding (9) and (9') and dividing by 2, we obtain (5).

Theorem A can be extended to measures defined for Borel subsets of the entire real line R. Let $\mathfrak{B}(R)$ be the smallest family of subsets of R containing all closed subsets of R and closed under the formation of countable unions and complements relative to R. By a measure on $\mathfrak{B}(R)$, we mean a nonnegative, finite, real-valued function, say μ, defined for all sets in $\mathfrak{B}(R)$, for which (3) holds for all pairwise disjoint sets $A_1, A_2, A_3, \cdots$ in $\mathfrak{B}(R)$. For $t\in R$, let us write

$$\begin{aligned} M(t) &= \tfrac{1}{2}\{\mu(]-\infty,t]) + \mu(]-\infty,t[)\}, \\ N(t) &= \tfrac{1}{2}\{\nu(]-\infty,t]) + \nu(]-\infty,t[)\}, \end{aligned} \tag{10}$$

where μ and ν are any measures on $\mathfrak{B}(R)$. The analogue for Theorem A dealing with measures on $\mathfrak{B}(R)$ can now be stated.

THEOREM B. *Let μ and ν be measures on $\mathfrak{B}(R)$ and M and N functions on R as defined in* (10). *Then*

$$\int_R M(t)d\nu(t) + \int_R N(t)d\mu(t) = \mu(R)\nu(R). \tag{11}$$

Proof. For every positive integer k and every real number t, let $M_k(t) = \frac{1}{2}\{\mu([-k,t]) + \mu([-k,t[)\}$ if $|t| \leq k$ and $M_k(t) = 0$ if $|t| > k$. Let $N_k(t)$ be defined similarly, with μ replaced by ν. Theorem A shows that

$$\int_R M_k(t)d\nu(t) + \int_R N_k(t)d\mu(t) = \mu([-k,k])\cdot\nu([-k,k]). \tag{12}$$

The sequence of functions $\{M_k\}_{k=1}^{\infty}$ is increasing and has pointwise limit M; similarly for $\{N_k\}_{k=1}^{\infty}$. B. Levi's theorem on integration of monotone increasing sequences of integrable functions shows that

$$\begin{aligned} \lim_{k\to\infty} \int_R M_k(t)d\nu(t) &= \int_R M(t)d\nu(t), \\ \lim_{k\to\infty} \int_R N_k(t)d\mu(t) &= \int_R N(t)d\mu(t). \end{aligned} \tag{13}$$

The limit of the right side of (12) is clearly $\mu(R)\cdot\nu(R)$. Combining (12) and (13), we obtain (11).

Theorems A and B have obvious extensions to the case of *complex* measures defined on $\mathfrak{B}([a,b])$ and $\mathfrak{B}(R)$, respectively. A complex measure on $\mathfrak{B}(R)$, say, is a complex-valued function defined for all sets in $\mathfrak{B}(R)$ and satisfying (3). It can be proved that every complex measure μ on $\mathfrak{B}(R)$ has the form $\mu = \mu_1 - \mu_2$

$+i\mu_3-i\mu_4$, where μ_1, μ_2, μ_3, and μ_4 are measures on $\mathfrak{B}(R)$ (see [2, pp. 120–123]). From this, Theorem B follows for complex μ and ν by an obvious computation. Theorem A has a similar obvious extension to complex measures.

Formula (1) is obtained from (5) for complex measures as follows. Let f and g be absolutely continuous complex-valued functions defined on $[a, b]$, that is, functions which are the integrals of their derivatives. For sets A in $\mathfrak{B}([a, b])$, let $\mu(A)=\int_A f'(t)dt$ and $\nu(A)=\int_A g'(t)dt$. Then, since f and g are absolutely continuous, we have

$$M(t)=\int_a^t f'(x)dx=f(t)-f(a) \quad \text{and} \quad N(t)=g(t)-g(a) \qquad (a \leqq t \leqq b).$$

Furthermore, for every bounded measurable function ψ on $[a, b]$, we have $\int_{[a,b]}\psi(t)d\mu(t)=\int_a^b\psi(t)f'(t)dt$ and $\int_{[a,b]}\psi(t)d\nu(t)=\int_a^b\psi(t)g'(t)dt$. Equality (5) now gives us

$$\int_a^b [f(t)-f(a)]g'(t)dt+\int_a^b [g(t)-g(a)]f'(t)dt=[f(b)-f(a)][g(b)-g(a)],$$

which is (1) in a slight disguise.

Remark. As pointed out in the footnote† (p. 421), the special case of Fubini's theorem needed to prove Theorem A is elementary. Let h be a continuous real-valued function on $[a, b]\times[a, b]$, and let μ and ν be measures on $\mathfrak{B}([a, b])$. The function h can be arbitrarily uniformly approximated on $[a, b]\times[a, b]$ by polynomials $p(t, u)=\sum a_{kl}t^k u^l$ (Weierstrass's polynomial approximation theorem for two variables). It is trivial that

$$\int_a^b\int_a^b p(t, u)d\mu(t)d\nu(u)=\int_a^b\int_a^b p(t, u)d\nu(u)d\mu(t). \tag{14}$$

It is then easy to see that (14) holds with p replaced by h. The function ϕ defined in the footnote† (p. 421) is the pointwise limit of a bounded decreasing sequence of continuous functions. Thus (14) holds with p replaced by ϕ. This is just the equality (6). Of course Fubini's theorem in its full generality is a much stronger assertion than (6), and requires a much more complicated proof. See [4, p. 87, Theorem (9.10)].

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A MINIATURE THEORY OF LEBESGUE AREA

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The problem of showing the existence of a surface minimizing the parametric two-dimensional integral considered by McShane, Danskin, Sigalov and Cesari [9, 6, 12, 4] is closely connected with the theory of Lebesgue area. When we reduce this problem to the one-dimensional case, the results, though not new, are interesting and illustrate the methods of that theory [1, 2, 3].

Let p and q be two points in Euclidean space E, and D be the collection of absolutely continuous functions x on an interval $[a, b]$ into E such that $x(a)=p$ and $x(b)=q$. We are interested in minimizing $I(f, x)=\int_a^b f(x(t), x'(t))dt$ for all $x\in D$, where f belongs to the class F to be described later. Tonelli observed that Hilbert's compactness theorem for curves of bounded length would be sufficient to provide the desired minimum if it were known that I_f were lower semicontinuous, and he showed [13] that I_f has that property under conditions weaker than we have imposed upon f. Later Menger introduced such metrics and distances which would enable him to interpret the integral as a generalized length and obtained another proof of Tonelli's result [10].

It is this theorem which we are interested in. We wish to present another proof that I_f is lower semicontinuous, provided $f\in F$, by interpreting $I(f, x)$ as a length (it might be more appropriate to consider the integral as the energy of a curve in a suitable field) where the length of a small line segment depends upon its position and direction, as well as its geometric length. To this end we introduce a "length" L_f in much the same way that Lebesgue defined his area. That L_f is lower semicontinuous is immediate, but we must now show that $L(f, x)=I(f, x)$ if x is absolutely continuous. The proof of this equality will be obtained by means of a process which is used in Lebesgue area theory. (For our present problem it would be sufficient to restrict ourselves to the functional P_f to be defined later.)

Let E^* be the surface of the unit sphere in E. Then F is the collection of real-valued functions f on $E\times E$ satisfying

(a) f is uniformly continuous on $E\times E^*$,

(b) there exist numbers $M\geqq m>0$ such that

$$m\|p\| \leqq f(a, p) \leqq M\|p\|, \qquad a\in E, p\in E,$$

(c) if $k>0$ then $f(a, kp)=kf(a, p)$,

(d) if $f^a(p)=f(a, p)$, then f^a is convex for each $a\in E$.

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Let $C(J)$ be the space of continuous functions x on a closed interval J into E. For absolutely continuous functions x let $I(f, x) = \int_J f(x(t), x'(t))dt$.

DEFINITION. *If $x \in C(J)$ then $L(f, x) = \liminf_{z \to x} I(f, z)$ where $z \in C(J)$ is quasilinear and the arrow indicates uniform convergence.*

If $z \in C(J)$ is quasilinear, and if we put $z_n = z$ for $n = 1, 2, \cdots$, then $z_n \to z$ and we obtain $L(f, z) \leqq \liminf I(f, z_n) = I(f, z)$.

Next we use a diagonalization process to show that L_f is lower semicontinuous. Let $x_n \to x$ in $C(J)$. Then there exist quasilinear functions z_{n_k} in $C(J)$ such that $z_{n_k} \to x_n$ and $I(f, z_{n_k}) \to L(f, x_n)$. Thus there is an integer k_n such that $\|z_{n_{k_n}} - x_n\| < 1/n$ and $I(f, z_{n_{k_n}}) < L(f, x_n) + 1/n$. Hence $z_{n_{k_n}} \to x$ and $L(f, x) \leqq \liminf I(f, z_{n_{k_n}}) = \liminf L(f, x_n)$.

Now suppose that S is a lower semicontinuous functional on $C(J)$ such that $S(z) = I(f, z)$ for quasilinear z. If $x \in C(J)$ then there exists $z_n \in C(J)$, z_n quasilinear, such that $z_n \to x$ and $I(f, z_n) \to L(f, x)$. Thus $S(x) \leqq \liminf I(f, z_n) = L(f, x)$.

It remains for us to show that $L(f, x) = I(f, x)$ if x is absolutely continuous. At this stage we do not even know that the equality holds for quasilinear functions, though we have just seen that $L_f \leqq I_f$ on such functions. This problem also arises in the theory of Lebesgue area. It is customary to answer it by introducing what Radó called a "lower" area and studying the relation between the lower and the Lebesgue areas. We shall define a "lower" length P_f, also lower semicontinuous, in analogy to a particular lower area, Peano area. It should be noticed that L_f was defined globally and, conceivably, is too small on quasilinear functions, while P_f will be defined locally and, at first glance, may appear to be too big on quasilinear functions. We shall use Theorem 5, analogous to a theorem of Tonelli concerning Jordan length, to obtain $P(f, x) \geqq I(f, x)$ for rectifiable x with the equality holding if and only if x is absolutely continuous. In particular, $P(f, z) = I(f, z)$ if z is quasilinear. From this last equality and the lower semicontinuity of P_f we have $P_f \leqq L_f$. In Theorem 6 we get $L_f \leqq P_f$. Thus we obtain our desired result: $L(f, x) = P(f, x) = I(f, x)$ if x is absolutely continuous. Important questions concerning Lebesgue area can also be solved by showing that a lower area agrees with the Lebesgue area, but the establishment of this equality, as distinguished from the one-dimensional case we are considering, is distinctly nontrivial, and is generally considered to be the principal theorem in the theory of Lebesgue area [5, 8].

We begin our discussion by making use of the fact that a convex function can be described by means of its supporting linear functionals.

If $p^* \in E^*$, then we interpret p^* as a linear functional on E by putting $p^*(p) = [p, p^*]$, the scalar product of p and p^*, for all p in E.

THEOREM 1. *Let f be a real-valued function on $E \times E$. A necessary and sufficient condition that $f \in F$ is that there exist points p_i^* dense in E^*, $i = \pm 1, \pm 2, \cdots$, $p_i^* + p_{-i}^* = 0$, and an equicontinuous family $\{f_i\}$ of real-valued functions on E such that range $f_i \subset [m, M]$ and $f(a, p) = \sup_i f_i(a)[p, p_i^*]$.*

Proof. The sufficiency is not hard and we will only show necessity. If $p^* \in E^*$ and $a \in E$ then f^a and p^* are continuous functions on E^* and $f^a(p) \geqq m > 0$ for all $p \in E^*$. Thus p^*/f^a is also continuous on the compact set E^* and we can define

$$\frac{1}{\psi(p^*, a)} = \max_{p \in E^*} \frac{[p, p^*]}{f(a, p)}.$$

Let us notice that there exists $T(p^*, a) \in E^*$ such that $f(a, T(p^*, a)) = \psi(p^*, a)[T(p^*, a), p^*]$. Furthermore, $f(a, p) \geqq \psi(p^*, a)[p, p^*]$ for all $p \in E^*$ and, by (c), for all $p \in E$. Since $[p, p^*] \leqq 1$ for all $p \in E^*$ we have $1/\psi(p^*, a) \leqq 1/m$. On the other hand,

$$\frac{1}{\psi(p^*, a)} \geqq \frac{[p^*, p^*]}{f(a, p^*)} \geqq \frac{1}{M}.$$

Thus $m \leqq \psi(p^*, a) \leqq M$ for all $p^* \in E^*$ and $a \in E$. It follows that $[T(p^*, a), p^*] \geqq m/M$. Now choose $\epsilon > 0$. By hypothesis there exists $\delta > 0$, $\delta < m\epsilon/(2M^2)$, such that $|f(a, p) - f(b, q)| < m\epsilon/(2M)$ whenever $a, b \in E$, $p, q \in E^*$, $\|a - b\| < \delta$ and $\|p - q\| < \delta$. Now suppose that $a, b \in E$, $p^*, q^* \in E^*$, $\|a - b\| < \delta$, and $\|p^* - q^*\| < \delta$. Then

$$\begin{aligned}\{\psi(p^*, a) - \psi(q^*, b)\}[T(q^*, b), q^*] &= \psi(p^*, a)[T(q^*, b), p^*] \\ &\quad + \{\psi(p^*, a)([T(q^*, b), q^*] - [T(q^*, b), p^*])\} - \psi(q^*, b)[T(q^*, b), q^*] \\ &\leqq f(a, T(q^*, b)) + M\delta - f(b, T(q^*, b)) < \frac{m\epsilon}{2M} + \frac{m\epsilon}{2M} = \frac{m\epsilon}{M}.\end{aligned}$$

Thus $\psi(p^*, a) - \psi(q^*, b) < (m\epsilon/M) \cdot (M/m) = \epsilon$. Similarly $\psi(q^*, b) - \psi(p^*, a) < \epsilon$. Now let p_i^* be dense in E^*, $i = \pm 1, \pm 2, \cdots$, with $p_i^* + p_{-i}^* = 0$, and put $f_i(a) = \psi(p_i^*, a)$. We have just shown that $\{f_i\}$ is equicontinuous and range $f_i \subset [m, M]$. Now fix $a \in E$ and $p \in E^*$. By the Hahn-Banach Theorem there exists a linear functional ϕ^* on E such that $\phi^*(p) = f(a, p)$ and $\phi^*(q) \leqq f(a, q)$ for all $q \in E$. Since ϕ^* is a linear functional over E there exists $p^* \in E^*$ and a constant k such that $\phi^* = kp^*$. We compute

$$\frac{1}{\psi(p^*, a)} \geqq \frac{[p, p^*]}{f(a, p)} = \frac{[p, p^*]}{k[p, p^*]} = \frac{1}{k}$$

and conclude that $\psi(p^*, a) \leqq k$. On the other hand,

$$\frac{[q, p^*]}{f(a, q)} \leqq \frac{[q, p^*]}{\phi^*(q)} = \frac{[q, p^*]}{k[q, p^*]} = \frac{1}{k}$$

for all $q \in E^*$. Thus $1/\psi(p^*, a) = 1/k$. Hence $f(a, p) = \psi(p^*, a)[p, p^*]$. Next we compute $|\psi(p^*, a)[p, p^*] - \psi(p_i^*, a)[p, p_i^*]| \leqq |\psi(p^*, a) - \psi(p_i^*, a)|\|p\| + |\psi(p_i^*, a)|\|p^* - p_i^*\|$. Now let $\epsilon > 0$. Since $\{p_i^*\}$ is dense in E^*, and ψ, as we

have seen, is uniformly continuous on $E^*\times E$, there exists an integer j such that $|\psi(p^*, a)[p, p^*]-\psi(p_j^*, a)[p, p_j^*]|<\epsilon$. Thus $f(a, p)\leqq\psi(p_j^*, a)[p, p_j^*]+\epsilon$. Since we have already observed that $f(a, p)\geqq\psi(q^*, a)[p, q^*]$ for all $q^*\in E^*$ we conclude that $f(a, p)=\sup_i f_i(a)[p, p_i^*]$.

We now begin to duplicate some of the ideas used in studying Lebesgue area. Our immediate goal is to define the Peano length P_f, but to do this we require the notion of a flat map. A *flat map* is a continuous function on a closed interval J into E_1, where J and E_1 are both oriented. Let ϕ be such a function.

If U is an open interval in E_1, let $K(\phi, U)$ be the collection of components of $\{p\in J|\phi(p)\in U\}$. If $V\in K(\phi, U)$, define

$$D(\phi, U, V)=\begin{cases}1 & \text{if } \phi|V \text{ maps } V \text{ onto } U \text{ preserving orientation,}\\ -1 & \text{if } \phi|V \text{ maps } V \text{ onto } U \text{ reversing orientation,}\\ 0 & \text{if } \phi|V \text{ does not map } V \text{ onto } U.\end{cases}$$

Then $D(\phi, U, V)$ is the *degree* of the mapping $\phi|V$, and can be defined in such a way as to make sense for higher-dimensional flat mappings [7]. It follows from the uniform continuity of ϕ that there are only finitely many $V\in K(\phi, U)$ for which $D(\phi, U, V)\neq 0$. Furthermore, if $U_1\subset U_2$ and $D(\phi, U_2, W)\neq 0$, then there exists $V\in K(\phi, U_1)$ such that $V\subset W$ and $D(\phi, U_1, V)=D(\phi, U_2, W)$.

Let D^+ and D^- be the positive and negative parts of D, respectively, so that $D=D^+-D^-$.

We introduce the following notation: $\lambda=+$ or $\lambda=-$; $\mu=+$, $\mu=-$, or μ is to be disregarded. Thus $A^\mu=B^\mu$ means that $A^+=B^+$, $A^-=B^-$, and $A=B$. If U is a set then $c(U)$ is the characteristic function of U. We write $c(U, t)$ for $[c(U)](t)$. Finally, we put $f_i^\lambda=f_{\lambda i}$ (see Theorem 1).

Now let $x\in C(J)$ and define $x^i=[x, p_i^*]$. Evidently x^i is a flat map for each i.

We modify the degree of the mapping to take account of the fact that we wish our length to depend upon $f\in F$. In what follows, U is an open interval in E_1 and $V\in K(x^i, U)$. We define

$$O_i^\lambda(x, U, V)=\inf_{p\in V} f_i^\lambda(x(p))D^\lambda(x^i, U, V),$$

$$O_i=\overset{+}{O}_i+\overset{-}{O}_i.$$

The function M_i^n is defined on $C(J)\times E_1$ by

$$M_i^n(x, t)=\sup_U \sum_{V\in K(x^i, U)} O_i^n(x, U, V)c(U, t).$$

Let $N<M_i^n(x, t)$ and suppose that $t_n\to t$. Then there exists U such that $\sum_{V\in K(x^i, U)} O_i^n(x, U, V)c(U, t)>N$. Furthermore, if n is sufficiently large then $t_n\in U$, from which it follows that $\liminf M_i^n(x, t_n)>N$. Thus M_i^n is lower semi-continuous in its second argument. Since f_i^λ is uniformly continuous, it requires only a slight modification of Federer's argument to show that M_i^n is also lower

semicontinuous in its first argument.

Let $\bar{f}(a, p) = \sup_i [p, p_i^*]$. It is not hard to see that $\bar{f}(a, p) = \|p\|$ and that $\bar{f}_i \equiv 1$ for all i. The function M_i evaluated for $\bar{f}$ is a *multiplicity function* and, as defined here, follows very closely the definition of Federer, which is valid for flat maps of arbitrary finite dimension.

THEOREM 2. $M_i = M_i^+ + M_i^-$.

Proof. Let $N^\lambda < M_i^\lambda(x, t)$. Then there exists an open interval U^λ in E_1 such that

$$\sum_{V \in K(x^i, U^\lambda)} O_i^\lambda(x, U^\lambda, V) c(U^\lambda, t) > N^\lambda.$$

Now let $U = U^+ \cap U^-$. Then $U \subset U^\lambda$ and for each $V \in K(x^i, U^\lambda)$ there exists $V^\lambda \in K(x^i, U)$ such that $V^\lambda \subset V$ and $D(x^i, U, V^\lambda) = D(x^i, U^\lambda, V)$. Since $\inf_{p \in V^\lambda} f_i(x(p)) \geqq \inf_{p \in V} f_i(x(p))$, it follows that $\sum' O_i(x, U, V^\lambda) c(U, t) \geqq \sum'' O_i^\lambda(x, U^\lambda, V) c(U, t) > N^\lambda$ where $\sum''$ indicates summation over $V \in K(x^i, U^\lambda)$ and $\sum'$ indicates summation only over such $V^\lambda \in K(x^i, U)$ for which there exists $V \in K(x^i, U^\lambda)$ such that $V^\lambda \subset V$ and $D(x^i, U, V^\lambda) = D(x^i, U^\lambda, V)$. Thus the sum over all $W \in K(x^i, U)$ exceeds $N^+ + N^-$ and we get $M_i(x, t) \geqq N^+ + N^-$. Hence $M_i(x, t) \geqq M_i^+(x, t) + M_i^-(x, t)$. The opposite inequality is an immediate consequence of the definitions. The corresponding theorem for the two-dimensional problem asserts only that the equality holds except on a countable set.

DEFINITION. $W_i(x) = \int_{-\infty}^{\infty} M_i(x, t) dt$.

Where necessary we modify our previous notation to show that O, M, and W depend upon f. Thus we write $W_i(f, x)$ for $W_i(x)$.

THEOREM 3. *A necessary and sufficient condition that x^i be of bounded variation is that $W_i(f, x)$ be finite.*

Proof. Cesari, with definitions phrased slightly differently, and Federer showed that $W_i(\bar{f}, x) = L(x^i)$, the geometric length of the flat map x^i, [1, 7]. Since range $f_i \subset [m, M]$, it follows that

$$mO_i(\bar{f}, x, U, V) \leqq O_i(f, x, U, V) \leqq MO_i(\bar{f}, x, U, V),$$

where U is an open interval and $V \in K(x^i, U)$. Thus

$$mM_i(\bar{f}, x, t) \leqq M_i(f, x, t) \leqq MM_i(\bar{f}, x, t)$$

and, using the result just referred to, we get

$$mL(x^i) \leqq W_i(f, x) \leqq ML(x^i).$$

We can interpret $W_i(x)$ and $L(x^i)$ as interval functions by putting $[W_i(x)](K) = W_i(x | K)$ and $[L(x^i)](K) = L(x^i | K)$ for closed intervals $K \subset J$. We recall that

an additive function of bounded variation has a derivative and that $L(x^i)' = |x^{i\prime}|$ almost everywhere. It is easy to verify that $W_i(x)$ is additive.

THEOREM 4. *Let x^i be of bounded variation. Then*

$$W_i(x)' = \max\,(f_i^+(x)x', -f_i^-(x)x')$$

almost everywhere.

Proof. Since $W_i(x)$ and x^i are both of bounded variation, the indicated derivatives exist almost everywhere. Suppose they exist at p, and that $x^{i\prime}(p) \geqq 0$. Let $\epsilon > 0$. There is a closed interval $K \subset J$ centered at p such that $|f_i(x(p)) - f_i(x(q))| < \epsilon$ provided $q \in K$. Hence, as in the last theorem,

$$[f_i(x(p)) - \epsilon]L(x^i \mid K) < W_i(x \mid K) < [f_i(x(p)) + \epsilon]L(x^i \mid K).$$

Thus

$$[f_i(x(p)) - \epsilon]x^{i\prime}(p) \leqq W_i'(x, p) \leqq [f_i(x(p)) + \epsilon]x^{i\prime}(p).$$

The case $x^{i\prime}(p) < 0$ is handled similarly and the theorem follows.

DEFINITION. *If $x \in C(J)$ then $P(f, x) = \sup_\pi \sum_{I\in\pi} \sup_i W_i(x \mid I)$ where π is a finite set of nonoverlapping closed intervals contained in J.*

It follows from the lower semicontinuity of M that P_f is also lower semicontinuous.

LEMMA. *If the interval functions S_i, $i = 1, 2, \cdots, n$, all have derivatives at p and if $T = \max S_i$, then T has a derivative at p and $T'(p) = \max S_i'(p)$.*

Proof. Let $\underline{T}$ and $\overline{T}$ be the lower and upper derivates of T. Since $T(I) \geqq S_i(I)$ for all intervals I and each i, we obtain $\underline{T}(p) \geqq \max S_i'(p)$. Now choose $\{I_k\}$ so that $p \in I_k$, $(\text{meas } I_k) \to 0$, and $T(I_k)/(\text{meas } I_k) \to \overline{T}(p)$. Then for some $i = i_0$ there exists a subsequence $\{I_{k_m}\}$ with $(\text{meas } I_{k_m}) \to 0$ and $T(I_{k_m}) = S_{i_0}(I_{k_m})$ for all m. Thus $\overline{T}(p) = S_{i_0}(p) \leqq \max S_i'(p) \leqq \underline{T}(p)$.

THEOREM 5. *If x is rectifiable then $P(f, x)' = f(x, x')$ almost everywhere. Thus $P(f, x) \geqq I(f, x)$ and the equality holds if and only if (each component of) x is absolutely continuous. In particular, $P(f, z) = I(f, z)$ if z is quasilinear.*

Proof. Let $T_n(I) = \max_{1\leq i\leq n} W_i(x \mid I)$, $T(I) = \sup_i W_i(x \mid I)$. Then $P(f, x) = \text{Variation } T$ and $\{\text{Variation } T_n\}$ is monotonically increasing with limit (Variation T). Thus [11, p. 116, 121], (Variation T)$' = \lim$ (Variation T_n)$' = \lim |T_n'| = \sup W_i'$ almost everywhere.

Let p_i^*, $i = 1, 2, \cdots, N$, be a basis for E consisting of points of E^* all of whose coordinates are zero except one. (It may be that the points have to be relabelled to satisfy the above condition.) Since f^a is convex for each $a \in E$, we can use the proof of Theorem 3 to get

$$mL(x^i) \leqq W_i(x) \leqq P(f, x) \leqq \sum W_i(x) \leqq M \sum L(x^i).$$

Thus $P(f, x)$ is absolutely continuous if and only if $L(x^i)$ is absolutely continuous for each i. Hence $P(f, x)$ is absolutely continuous if and only if x^i is absolutely continuous for each i. The remainder of the proof now follows from [11, p. 119].†

THEOREM 6. *If $x \in C(J)$ there exists a sequence $\{z_n\}$ of quasilinear functions inscribed in x such that $z_n \to x$ and $I(f, z_n) \to P(f, x)$. Thus $P_f = L_f$.*

Proof. Let π_n be a partition of J into intervals whose maximum diameter is less than $1/n$ and such that $|f(b, p^*) - f(a, p^*)| < m/n$ for all $p^* \in E^*$ whenever $\|b-a\| \leqq \max \|x(u)-x(t)\|$ for all u, t in an interval of π_n. Let $z_n \in C(J)$ be the quasilinear function inscribed in x whose intervals of linearity are the elements of π_n. For each i and each $K \in \pi_n$, $M_i(z_n | K, t) < (m+m/n)m^{-1}M_i(x | K, t)$. Hence $I(f, z_n) = \sum I(f, z_n | K) = \sum P(f, z_n | K) < (1+1/n) \sum P(f, x | K) = (1+1/n)P(f, x)$. Thus $L(f, x) \leqq \liminf I(f, z_n) \leqq P(f, x) \leqq L(f, x)$.

It follows that I_f is lower semicontinuous with respect to uniform convergence on the class of absolutely continuous functions. Thus we have another proof of the Tonelli theorem.

Our discussion would be incomplete without some mention of a "curve." One reason for the importance of a *Fréchet curve* is that it allows us to use arc-length to parametrize a rectifiable curve [2]. An easy argument shows that L_f has the same value on all Fréchet equivalent functions, and is lower semicontinuous with respect to Fréchet convergence.

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† The statements in [11] refer to additive interval functions, and we require the less restricted functions of Banach. The theorems hold for these functions too. See page 116.

ALMOST DIAGONAL MATRICES

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1. Introduction. In the following, capital English letters will denote $N \times N$ matrices with real elements, lower case English letters will denote vectors, and A^t, x^t denote the transposed matrix A and vector x, respectively. We use (x, y) for the inner product of x and y.

DEFINITION. *A matrix A is almost diagonal (a.d.) if there exist a diagonal matrix D and vectors x and y such that*

$$A = D + xy^t. \tag{1}$$

That is, A is a.d. if it differs from a diagonal matrix by a matrix of rank one. The class of a.d. matrices is of importance in many applications. The following result is well known [1]:

LEMMA 1. *If D is nonsingular and*

$$(y, D^{-1}x) \neq -1 \tag{2}$$

then A is nonsingular and

$$A^{-1} = D^{-1}\left[I - \frac{xy^tD^{-1}}{1 + (y, D^{-1}x)}\right]. \tag{3}$$

2. The diagonal form. Suppose A is a.d., and ξ is an eigenvector of A corresponding to the eigenvalue λ. Then

$$\lambda\xi_i = \sum A_{ij}\xi_j = D_{ii}\xi_i + (y, \xi)x_i. \tag{4}$$

If (y, ξ) does not vanish, we may take $(y, \xi) = 1$ for normalization, and find

$$\xi_i = \frac{x_i}{\lambda - D_{ii}}, \tag{5}$$

and substituting back into (4) we find

$$\phi(\lambda) \equiv \sum_{i=1}^{N} \frac{x_iy_i}{\lambda - D_{ii}} = 1. \tag{6}$$

Thus the eigenvalues of A corresponding to eigenvectors which are not orthogonal to y satisfy (6). Now (6) fails to have N roots if either an x_i vanishes, or a y_i vanishes, or two or more of the D_{ii} are repeated, in any of which cases it is easy to see that (6) reduces to a polynomial equation of degree less than N

after suitable clearing. Now if a y_i vanishes for some i, then y is orthogonal to the eigenvector of D for the eigenvalue D_{ii} and (4) shows that that same vector is an eigenvector of A with the same eigenvalue. Similarly if an x_i vanishes, we may argue with A^t to the same conclusion that D_{ii} is an eigenvalue of A. Finally, if D_{ii} is a repeated diagonal element of D, say p times, then there are at least $p-1$ independent eigenvectors of D which are orthogonal to y, and all of these are eigenvectors of A with the eigenvalue D_{ii}. If p eigenvectors of D are orthogonal to y, then p components of y vanish, and we are back to the previous case. Summarizing, we have

LEMMA 2. *The roots of* (6) *are the eigenvalues of* A. *If a term fails to appear in* (6) *because its numerator vanishes, then the corresponding element of* D *is an eigenvalue of* A. *If* p *terms of* (6) *all have the same denominator, so that one can collect terms, then the corresponding element of* D *is a* $(p-1)$*-fold eigenvalue of* A. *All of the eigenvalues of* A *are accounted for by the above.*

In the following we shall assume that no components of x or y vanish.

Now suppose the D_{ii} are distinct, and let π be the permutation of the integers $1, 2, \cdots, N$ such that

$$D_{\pi(1),\pi(1)} < D_{\pi(2),\pi(2)} < \cdots < D_{\pi(n),\pi(n)}. \tag{7}$$

Then for any $\alpha = 1, 2, \cdots, N$,

$$\phi(D_{\alpha\alpha} + \epsilon) = \frac{x_\alpha y_\alpha}{\epsilon} + O(1) \qquad (\epsilon \to 0), \tag{8}$$

$$\phi(O_{\alpha\alpha} - \epsilon) = -\frac{x_\alpha y_\alpha}{\epsilon} + O(1) \qquad (\epsilon \to 0), \tag{9}$$

and we have

LEMMA 3. *Suppose the* D_{ii} *are distinct, and that for some* k,

$$\operatorname{sgn}(x_{\pi(k)} y_{\pi(k)}) = \operatorname{sgn}(x_{\pi(k+1)} y_{\pi(k+1)}). \tag{10}$$

Then an eigenvalue of A *lies between* $D_{\pi(k),\pi(k)}$ and $D_{\pi(k+1),\pi(k+1)}$.

Clearly if A is symmetric, (10) always holds. In any case

LEMMA 4. *If*

$$\operatorname{sgn}(x_i y_i) = \textit{const.} \qquad (i = 1, 2, \cdots, N) \tag{11}$$

and the D_{ii} *are distinct, then all the eigenvalues of* A *are real, distinct, and separated by the* D_{ii}.

Letting $\lambda_1, \lambda_2, \cdots, \lambda_n$ denote the roots of (6), the matrix

$$P_{ij} = \frac{x_i}{\lambda_j - D_{ii}} \qquad (i, j = 1, 2, \cdots, N) \tag{12}$$

by (5) is the matrix whose columns are the eigenvectors of A, and if the λ_j are distinct, we have

$$A = P\Lambda P^{-1}, \tag{13}$$

with

$$\Lambda_{ij} = \lambda_i \delta_{ij}. \tag{14}$$

3. The matrix P^{-1}:

Our purpose now is to exhibit the elements of P^{-1} in closed form.

LEMMA 5. *Suppose the eigenvalues of A are distinct. Then we have*

$$(P^{-1})_{ij} = \frac{1}{(\lambda_i - D_{jj})\psi'(\lambda_i)}, \tag{15}$$

where

$$\psi(z) = 1 - \sum_{i=1}^{N} \frac{x_i y_i}{z - D_{ii}}. \tag{16}$$

To prove this, we make use of the Cauchy integral formula in matrix form [2], namely

THEOREM 2. *Let M be an $N \times N$ matrix with distinct eigenvalues. Let C be a simple, closed contour in the complex plane which encloses a domain Δ. Suppose that all the eigenvalues of M lie in Δ, and let $f(z)$ be a function regular in Δ. Then*

$$f(M) = \frac{1}{2\pi i} \oint_C \frac{f(z)dz}{(zI - M)}. \tag{17}$$

Now let A be a.d. with distinct eigenvalues, and take $f(z) = 1$ in the above theorem. Then

$$I = \frac{1}{2\pi i} \oint_C (zI - A)^{-1} dz. \tag{18}$$

Clearly $zI - A$ is also a.d., and by (3),

$$(zI - A)^{-1} = (zI - D)^{-1} + \frac{1}{\psi(z)} (zI - D)^{-1} x y^t (zI - D)^{-1}. \tag{19}$$

We see that the poles of the integrand are at the zeros of $\psi(Z)$, that is, at the λ_i, and that if C has been deformed to exclude the D_{ii}, the first term of (19) contributes nothing to (18).

Further,

$$\operatorname{Res} \left\{ (zI - A)^{-1}_{ij} \right\}_{z=\lambda_p} = \frac{x_i y_j}{(\lambda_p - D_{ii})(\lambda_p - D_{jj})\psi'(\lambda_p)},$$

and (18) gives

$$\delta_{ij} = \sum_{p=1}^{N} \frac{x_i y_j}{(\lambda_p - D_{ii})(\lambda_p - D_{jj})\psi'(\lambda_p)} = \sum_{p=1}^{N} P_{ip}(P^{-1})_{pj}. \tag{20}$$

Comparison with (12) shows that (15) and (16) hold, which completes the proof of Lemma 5.

4. Summary. We summarize the preceding developments with

THEOREM 2. *Let A be a.d. with distinct eigenvalues, and let $f(z)$ be regular in some region containing all the eigenvalues of A. Then*

$$f(A) = Pf(\Lambda)P^{-1}, \tag{21}$$

where P is given explicitly by (12), *P^{-1} by* (15), (16), *and Λ by* (14) *where the λ_i satisfy* (6).

Theorem 2 permits, for example, the explicit solution of a system of linear ordinary differential equations with an a.d. coefficient matrix.

We conclude with

THEOREM 3. *Let A be a.d. and symmetric, so that*

$$A = D + \sigma x x^t. \tag{22}$$

Suppose the elements of D are distinct and that no x_i vanishes. Then, necessary and sufficient conditions that A be nonnegative definite are

$$1^\circ \sum_{i=1}^{N} \frac{x_i^2}{D_{ii}} \leqq -\frac{1}{\sigma} \text{ and } 2^\circ\ D_{11} \geqq 0, \text{ if } \sigma \leqq 0;$$

$$1'\ D_{22} \geqq 0 \text{ and either } 1^\circ \text{ or } 2^\circ, \text{ if } \sigma > 0.$$

(In the above, the D_{ii} have been renumbered in ascending algebraic order.)

The proof follows by inspection of the characteristic equation.

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A LOOK AT MATHEMATICAL COMPETITIONS IN HUNGARY

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The readers of this MONTHLY may be interested in some up-to-date information on mathematical competitions in Hungary. Contrary to the implication in the recent article by R. C. Buck [1], the Eötvös prize competition did not terminate in 1928. It is, in fact, still held annually (except for the years 1919–1921, 1944–1946, 1956) so that in the autumn of 1960, the sixtieth competition will be held. Besides the winners mentioned in [1], one may also mention, T. Gallai, L. Kalmár, D. König, F. Lukács, G. Sándor, M. Schweitzer, T. Szele, E. Teller, L. Tisza, and E. Vázsonyi, so that the relationship between this contest and mathematical fertility in Hungary seems to continue. Since 1949, the mathematical contest has been called the Kürschák prize competition, and the Eötvös competition limited to physics—certainly a more suitable choice.

Information about the contest can be found in issues of the journal *Középiskolai Matematikai Lapok* [Mathematical Journal for High Schools], and in the book [2] by Kürschák, mentioned in [1], which was re-edited in 1955 and to which a second volume was added in 1957 [3].

It should be said that the contest is now open to younger students than high school graduates. Indeed, 12 such contestants have won prizes since 1947. It might be of interest to give here the problems on the 1958 contest [4]. There were a total of 417 contestants, of whom only 317 handed in their work.

1. Consider six points in the plane of which no three are collinear. Show that three may be chosen to form a triangle in which one angle is greater than or equal to 120°.

2. If u and v are integers such that u^2+uv+v^2 is a multiple of 9, show that u and v must both be multiples of 3.

3. In the convex hexagon $ABCDEF$, suppose that every pair of opposite sides are parallel. Prove that the triangles ACE and BDF have equal areas.

Contestants have five hours to work on three problems. They may use books or other references. Stress is laid on ability and insight, with bonus points given for generalization. Problems are proposed by research mathematicians. In an evaluating lecture, many possible solutions and generalizations are given. The contest is administered by the sections of the J. Bolyai Mathematical Society (analogous, I believe, to a combination of the MAA and the AMS, in the United States).

The reader might be interested in other mathematical contests at present held in Hungary. Describing these by the admission age of contestants, there are:

Age:	
15	Dániel Arany prize for beginners.
16	Dániel Arany prize for advanced.
17–18	National Students Competition.

(Dániel Arany was founder in 1894 and first editor of our high school mathematics journal.) These contests began in 1947. At the present time, all are two-stage contests, each round consisting of three questions. The first round is held in schools, and the second at central locations in the 19 divisions of the country. In 1958, there were respectively 2145, 1666, 1951 entrants to the first round and 189, 79, 323 contestants admitted to the second round, in the three contests. As an example, we give here the problems on the contest for beginners [5].

First round

1. Prove that any two numbers between 10 and 99 with the same tens digit can be multiplied thus: remove the units in the first factor and add to the second, then multiply the resulting numbers, and add to the result the product of the units in the original factors.

Example: $(23)\,(28) = (20)\,(31) + (3)\,(8) = 644$

2. In the right triangle ABC (with $A = 90°$) construct a circle with AC as diameter. Let it meet BC in E. Draw a tangent to this circle at E, meeting the remaining side in D. Show that EBD is an isoceles triangle.

3. An arc AD of a circle is divided into three equal arcs by the points B and C. Is the chord AD also divided equally by the radii to B and C?

Second round

1. For what integer values of b is $97b^3+84b-55$ a multiple of b?

2. Find the missing numbers (indicated by X) in the following division problem.

$$\begin{array}{r|l} & XX8 \\ \hline XXX & XXXXXX \\ & \underline{XXX5} \\ & XXXX \\ & \underline{9XX} \\ & XXX \\ & \underline{XXX} \\ & 0 \end{array}$$

3. Let Q be a fixed point that is not on the perimeter of a given square $ABCD$. For any choice of a point P on the perimeter, choose a point R so that PQR is an equilateral triangle. As P moves along $ABCD$, what is the path traced out by R?

In addition to these contests, the high school journal also conducts several problem-solving contests. There are usually 72 elementary and 72 advanced problems proposed in each year. In 1957–58, 1691 students participated, and were rewarded by "points", with winners receiving various prizes, last but not least of which is to have their photographs appear in the journal.

There also are contests for university students. Let me mention here only the Schweitzer Memorial Competition sponsored by the Mathematical Society. This is held in all universities (students of age 19–24 in general) with problems proposed by research mathematicians. As a rule, there are about 10 problems in multiple-choice form which are to be worked by the contestants at home during a ten-day period. Here also evaluating lectures used to be held in all uni-

versity seats. As an example, we give here some of the problems for the 1957 competition which was organized by the Debrecen section and given to 19 contestants [6].

1. Let (C_{ij}) be a real orthogonal matrix. Show that the mapping $w=f(z)=\sum_{i,j=1}^{3}C_{ij}z_i\bar{z}_j$ sends the surface $\sum_{i=1}^{3}z_i\bar{z}_i=1$ into a triangle.

3. Let A be a set of points in n-space which contains one interior point, and is such that if P and Q are in A, so is their mid-point. Show that A consists of a convex set and certain of its boundary points.

4. For $0<\epsilon<1$, let $\mathfrak{F}_\epsilon$ be the class of non-negative continuous functions $f(x)$ defined for $0\leq x<\infty$ and such that for all x_1, x_2

$$f(x_1)f(x_2)\leqq\epsilon^{|x_1-x_2|}.$$

Evaluate the expression

$$C(\epsilon)=\sup_{f\in\mathfrak{F}_\epsilon}\int_0^\infty f(x)dx.$$

5. Determine the real continuous functions satisfying the equation

$$f(xyz)=f(x)+f(y)+f(z)$$

(a) for all x, y, z different from 0,

(b) for all x, y, z in the interval $[a, b]$, where $1<a^3<b$.

7. Show that every real number x in the open interval (0, 1) can be written in the form $x=\sum_1^\infty 1/n_j$ where the n_j are positive integers, and n_{j+1}/n_j is always one of the integers 2, 3, 4.

8. Find all integers $c>1$ such that, for every prime p, the smallest positive number integer n with $c^n\equiv 1 \pmod{p}$ is different from 6.

10. We call an abelian group splitting if the subgroup consisting of elements of finite order is a direct summand. Show that an abelian group G is splitting if, for some integer m, the group mG, *i.e.*, {all mg for $g\in G$} is splitting.

In 1952, a competition was held for engineering and technological students, but was discontinued because their backgrounds seemed too disparate [7]. My personal opinion is that common problems could be found, and these contests continued also.

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AN EXISTENCE THEOREM FOR CERTAIN AUTONOMOUS SYSTEMS

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1. The usual existence proofs for ordinary differential equations use either the Picard method of successive approximations or the Cauchy method of sequences of approximate solutions, [1]. Both methods are very wide in scope but each one, by itself, is of local character. That is, the solution is shown to exist for a limited range of the independent variable.

In this note we consider a very narrowly restricted class of equations. For these we present an existence theorem whose proof uses neither of the above-mentioned methods and, in fact, it calls for nothing more advanced than the Weierstrass M-test, [2, p. 3]. Moreover, it yields the existence of analytic solutions for all real values of the independent variable. Finally, the method can be used, either by itself or in conjunction with Picard's or Cauchy's, on certain integral, integrodifferential and other functional equations, [3], [4]. However, it must be added that one can easily prove by Picard's method a result considerably more general than Theorem 1 below.

2. All variables, functions, and constants in the sequel are real. Accordingly, a function is everywhere analytic if for any real s_0 it can be expanded in a power series convergent in some interval about s_0. A system of differential equations is called autonomous, [1], if the independent variable does not occur in it explicitly. A first integral of a system E of differential equations, [5], is any equation which contains an arbitrary constant, is satisfied by the unknowns of E, and in which at least one of them occurs to lower derivative order than in E. For example, the autonomous system

$$x'' = -xy^2, \qquad y'' = -yx^2, \tag{1}$$

where $x=x(s)$, $y=y(s)$ and accents denote differentiation, has the first integral

$$x'^2 + y'^2 + x^2y^2 = c^2, \tag{2}$$

obtained by multiplying the first equation by x', the second one by y', adding the results and integrating the sum. System (1) occurs in the isoperimetric problem of maximizing the volume of the convex hull of a closed rectifiable space curve whose length is kept fixed, [6].

We consider the following autonomous system of ordinary differential equations together with initial conditions:

$$\begin{aligned} x_j^{(K_j)} &= f_j(x_1, x_2, \cdots, x_J, x_1', \cdots, x_J', x_1^{(p_j)}, \cdots, x_J^{(p_j)}), \\ x_j^{(k_j)} &= \xi_{jk_j}, \quad k_j = 0, 1, \cdots, K_j - 1, \ j = 1, 2, \cdots, J. \end{aligned} \tag{3}$$

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Here K_j is a positive integer, p_j a nonnegative one and $K_j > p_j$. The J unknowns $x_1, \cdots, x_J$ are functions of the independent variable s which is usually omitted, and upper indices denote differentiation with respect to s. The system (3) is called a polynomial one if each f_j is a polynomial in the indicated variables. It is then assumed that no polynomial f_j vanishes identically and that each f_j involves at least one term with a derivative of order p_j. Henceforth a system will mean a polynomial system. Let one of the monomial terms of f_j be

$$A[x_1^{(a_1)}]^{b_1}[x_1^{(a_2)}]^{b_2}\cdots[x_2^{(c_1)}]^{d_1}[x_2^{(c_2)}]^{d_2}\cdots, \tag{4}$$

where A is a constant and $0 \leqq a_1 < a_2 < \cdots$, $0 \leqq c_1 < c_2 < \cdots$; $b_1, b_2, \cdots$, $d_1, d_2, \cdots \geqq 1$. With this term we associate the weight

$$W = b_1(a_1+1) + b_2(a_2+1) + \cdots + d_1(c_1+1) + \cdots.$$

Let W_j be the maximum of the weights of the terms of f_j.

The system (3) is called bounded if it possesses a first integral, or a system of first integrals, implying the existence of majorizing functions ϕ_j, such that

$$|x_j^{(K_j-1)}(s)| \leqq \phi_j(|s|), \qquad j = 1, 2, \cdots, J, \tag{5}$$

where each $\phi_j(u)$ is a nonnegative nondecreasing continuous function for all $u \geqq 0$.

We shall prove

THEOREM 1. *A bounded polynomial system which satisfies the weight condition*

$$K_j \geqq W_j - 1, \qquad j = 1, 2, \cdots, J, \tag{6}$$

has a solution, necessarily unique, which is analytic for all s.

An example to which Theorem 1 applies is the system (1) with arbitrary initial conditions.

3. We assemble now some preliminary estimates and lemmas. If p and q are nonnegative integers we shall abbreviate $(p+q)!/p!$ to $[p]_q$. This will be used in the following context: if $x = \sum_0^\infty a_n s^n$ then $x^{(K)} = \sum_0^\infty a_{n+K}[n]_K s^n$.

LEMMA 1. *Let the constants $W, a_1, a_2, \cdots, b_1, \cdots, c_1, \cdots$ satisfy the conditions listed under* (4) *and let $K \geqq W-1$. Then*

$$[N]_K \geqq S = \sum [n_1]_{a_1}[n_2[_{a_1}\cdots[n_{b_1}]_{a_1}[m_1]_{a_2}[m_2]_{a_2}\cdots[m_{b_2}]_{a_2}[p_1]_{c_1}\cdots, \tag{7}$$

where $N = 0, 1, \cdots$ and the sum S is taken over all nonnegative solutions of $n_1+n_2+\cdots+m_1+m_2+\cdots+p_1+\cdots = N$, permutations counted distinct.

It is first verified that S is the coefficient of s^N in the power series of

$$F(s) = [f^{(a_1)}]^{b_1}[f^{(a_2)}]^{b_2}\cdots[f^{(c_1)}]^{d_1}\cdots,$$

where $f(s) = 1/(1-s)$. Since $f^{(a)} = a!/(1-s)^{a+1}$ and $[f^{(a)}]^b = (a!)^b/(1-s)^{b(a+1)}$,

$$S = (a_1!)^{b_1}(a_2!)^{b_2} \cdots [N]_{W-1}/(W-1)!.$$

By the hypothesis $K \geqq W-1$ and in order to prove (7) it suffices to show that $(a_1!)^{b_1}(a_2!)^{b_2} \cdots \leqq [b_1(a_1+1)+b_2(a_2+1)+ \cdots -1]!$. Since $b_1, b_2, \cdots \geqq 1$, the latter is clear and (7) is proved.

We define now the index I_j of the polynomial f_j to be the sum of the absolute values of the coefficients of f_j.

LEMMA 2. *In proving Theorem* 1 *it may be assumed without loss of generality that the indices I_j of the f_j's in* (3) *satisfy the condition*

$$0 < I_j \leqq 1, \qquad j = 1, 2, \cdots, J. \tag{8}$$

This is obvious since a linear transformation $s \to \lambda t$ of the independent variable achieves the purpose. Condition (6) guarantees then that the new indices can be made arbitrarily small. The condition of boundedness still holds for the new system.

From the hypothesis of boundedness, expressed by (5), we get by an easy induction on K

$$\begin{aligned} |x_j^{(K_j+K)}(s)| &\leqq \int_0^{|s|} \cdots \int_0^u \int_0^u \phi_j(u)dudu \cdots du \\ &\quad + \sum_{q=2}^{K} |\xi_{jK_j-q}| \, |s|^{K-q}/(K-q)!, \end{aligned}$$

where $K = 2, 3, \cdots, K_j$. Using the standard transformation of the above $(K-1)$-tuple integral we get

$$\begin{aligned} |x_j^{(K_j+K)}(s)| &\leqq 1/(K-2)! \int_0^{|s|} (|s| - u)^{K-2} \phi_j(u)du \\ &\quad + \sum_{q=2}^{K} |\xi_{j\,K-q}| \, |s|^{K-q}/(K-q)! = \phi_{jK}, \end{aligned} \tag{9}$$

where we also define ϕ_{j1} to be ϕ_j. All the majorants $\phi_{jK}(u)$ are finite nonnegative nondecreasing functions for $u \geqq 0$.

4. Assume the following power series expansions

$$x_j(s) = \sum_{n=0}^{\infty} \alpha_{nj} s^n, \qquad j = 1, 2, \cdots, J. \tag{10}$$

Substituting these and their derivatives into (3) and comparing the coefficients of s^N we have

$$[N]_{K_j} \alpha_{N+K_j\,j} = \sum Q_r, \tag{11}$$

where on the right there is a finite number of summands, each one contributed

by a single term of f_j. Consider now the sample term (4) of f_j—the corresponding Q_{r_1} term is

$$Q = AS\alpha_{n_1+a_1\,1}\alpha_{n_2+a_1\,1}\cdots\alpha_{nb_1+a_1\,1}\alpha_{m_1+a_2\,1}\alpha_{m_2+a_2\,1}\cdots\alpha_{p_1+a_1\,2}\cdots, \tag{12}$$

where the numerical factor S is the sum in (7). Let

$$R_1 = \max\,(1, \,|\,\xi_{jk_j}\,|\,/k_j!), \tag{13}$$

the maximum being taken over $k_j=0, 1, \cdots, K_j-1, j=1, 2, \cdots, J$; (13) is equivalent to

$$|\,\alpha_{k_j\,j}\,| \leqq R_1 \leqq R_1^{k_j+1}. \tag{14}$$

Suppose now that (14) holds for $k_j \leqq N+K_j-1, j=1, 2, \cdots, J$. Then by (11)

$$[N]_{K_j}\,|\,\alpha_{N+K_j\,j}\,| \leqq \sum |\,Q_r\,|, \tag{15}$$

and by (12), Lemma 1, the induction assumption and the weight condition (6) we have

$$|\,Q_{r_1}\,| \leqq [N]_{K_j}\,|\,A\,|\,R_1^{N+W} \leqq [N]_{K_j}\,|\,A\,|\,R_1^{N+K_j+1}.$$

Therefore by (8) and (15)

$$|\,\alpha_{N+K_j\,j}\,| \leqq R_1^{N+K_j+1}\sum|\,A\,| = I_jR_1^{N+K_j+1} \leqq R_1^{N+K_j+1}.$$

This completes the induction and therefore

$$|\,\alpha_{kj}\,| \leqq R_1^{k+1}, \quad k = 0, 1, \cdots; j = 1, 2, \cdots, J.$$

It follows that the power series (10) converge for $0 \leqq s < s_1 = 1/R_1$.

We choose now a small positive ϵ_1 and repeat the power series procedure at $s_1-\epsilon_1$ obtaining a continuation of the solutions $x_1(s), \cdots, x_J(s)$ from $0 \leqq s < s_1$ to $0 \leqq s < s_1+s_2-\epsilon_1$, where $s_2=1/R_2$ and R_2, like R_1, is given by

$$R_2 = \max\,(1, \,|\,x_j^{(k_j)}(s_1-\epsilon_1)\,|\,/k_j!),$$

the maximum being taken as in (13). Similarly, take a small $\epsilon_2>0$ and repeat the process at $s_1+s_2-\epsilon_1-\epsilon_2$, and so on. In n steps one obtains a continuation of the original power series (10) to the interval

$$0 \leqq s < \sum_{i=1}^{n} s_i - \sum_{i=1}^{n-1}\epsilon_i,$$

where $s_i=1/R_i$ and

$$R_i = \max\left(1, \left|\,x_j^{(k_j)}\left(\sum_{h=1}^{i-1}s_h - \sum_{h=1}^{i-1}\epsilon_h\right)\right| \Big/ k_j!\right).$$

Now by the estimates (9), following from the boundedness hypothesis,

$$R_i \leqq \max\left(1, \phi_{j\,K_j-k_j}\left(\sum_{h=1}^{i-1} s_h\right) \Big/ k_j!\right) \tag{16}$$

since the majorants ϕ_{jK} are nondecreasing. The sum $\sum_1^n \epsilon_i$ can be made uniformly small and by (16) the sum $\sum_1^n s_i$ must diverge with n since otherwise one of the majorants $\phi_{jK}(u)$ would have become infinite for some finite positive u, which is a contradiction. It follows that the solutions $x_j(s)$ of (3) are analytic for $s \geqq 0$. Their analyticity for $s \leqq 0$ is proved in exactly the same way and their uniqueness follows from the analyticity.

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MATHEMATICAL NOTES

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ON THE PROOF OF MERGELYAN'S APPROXIMATION THEOREM

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During the past year, the writer had the occasion to give an exposition in a seminar of the proof of the well-known theorem of Mergelyan [2], which states that a necessary and sufficient condition on a compactum K in the complex z-plane in order that every function f continuous on K and analytic in the interior of K should be uniformly approximable on K by polynomials in z, is that K should not disconnect the plane. Mergelyan's proof of this theorem, as given in [2] and Chapter 1 of [3], is quite elementary except at one point, where he makes use of a result for whose proof he refers to the general theory of capacity of planar sets. At first glance, this step does not appear to have the same elementary character as the rest of the proof. Upon closer examination, however, the writer found it possible to construct a direct and simple proof of the needed result. The method of proof has a superficial resemblance to the proof of the

"Flächensatz" in Bieberbach [1], pages 72–73, but argues on arc length rather than area.

PROPOSITION. *Let* $z=F(w)=aw+b+\sum_{j\geq 1} a_j w^{-j}$, *be a holomorphic function of* w *for* $|w|>1$ *with a simple pole at infinity. Suppose that* F *is continuous for* $|w|\geqq 1$, *that* F *maps the set* $|w|>1$ *into the complement of a compact subset* S *of the* z*-plane, and that* F *maps* $|w|=1$ *into the disk* $|z|\leqq 1$. *Suppose further that* S *has diameter* $\geqq k$, *for some positive number* k. *Then*

(α) $|a|\leqq 1$, $|b|\leqq 1$, $|a_j|\leqq 1$ *for all* $j\geqq 1$;

(β) $|a|\geqq g(k)>0$, *where* $[g(k)]^2=\max_{0<u<1}[k^2\pi^{-2}u-u^2(1+u)(1-u)^{-3}]$.

Proof. We begin with some elementary topological remarks.

(1) Since F maps the set $|w|=1$ into the disk $|z|\leqq 1$, and since F is continuous for $|w|\geqq 1$, given $\epsilon>0$, F must map the circle $C_r=\{w: |w|=r\}$ into the disk $|z|\leqq 1+\epsilon$ for $r>1$ and sufficiently close to 1.

(2) Let C_r' be the image of C_r under F. If z is a point in the complement of C_r' and C_r' is oriented in concordance with the usual positive direction on C_r, $0(z, r)$, the order of C_r' with respect to z, is well defined. For z in S, $0(z, r)$ is defined for all $r>1$ and hence independent of r. Since for r very large, $0(z, r)$ is different from zero, it must be nonzero for all $r>1$ for every z in S.

(3) Let z_0 and z_1 be two points of S with $|z_1-z_0|\geqq k$. For all real t, we define $z_t=z_0+t(z_1-z_0)$. We assert that there must exist t_1 and t_2, $t_1<0$ and $t_2>1$, such that z_{t_1} and z_{t_2} lie in C_r'. (Otherwise, $0(z_t, r)$ would be well defined and constant for $t\leqq 0$ and $t\geqq 1$. But $0(z_t, r)=0$ for $|t|$ sufficiently large, while $0(z_0, r)$ and $0(z_1, r)$ are different from zero and we would have a contradiction.) Obviously $|z_{t_1}-z_{t_2}|>k$, and the diameter of C_r' is greater than k. But C_r' is a rectifiable curve for $r>1$, and for a closed rectifiable curve, the arc length is at least twice the diameter. Thus, if $L(C_r')$ is the length of C_r', $L(C_r')>2k$ for all $r>1$.

Proof of (α). Let $r>1$, j an integer $\geqq -1$. Then,

$$\frac{1}{2\pi i}\int_{C_r} F(w)w^{j-1}dw=\begin{cases} a & j=-1,\\ b & j=0,\\ a_j & j\geqq 1.\end{cases}$$

Since $|F(re^{i\phi})|\leqq(1+\epsilon)$ for r sufficiently close to 1, it follows that $|a|\leqq(1+\epsilon)r^{-1}$, $|b|\leqq(1+\epsilon)$, $|a_j|\leqq(1+\epsilon)r^j$ for all r sufficiently close to 1, whence $|a|\leqq 1$, $|b|\leqq 1$, $|a_j|\leqq 1$.

Proof of (β). By the conclusion of Remark 3, above, we have

$$\text{(I)}\quad 4k^2r^{-2}\leqq r^{-2}[LC_r')]^2=r^{-2}\left(\int_0^{2\pi}|F'(re^{i\phi})|\,rd\phi\right)^2\leqq 2\pi\int_0^{2\pi}|F'(re^{i\phi})|^2d\phi.$$

We note that

$$\text{(II)} \qquad F'(re^{i\phi}) = a - \sum_{j\geq 1} j a_j r^{-j-1} e^{-(j+1)\phi}.$$

Applying the Parseval equality to (II), we obtain

$$\text{(III)} \qquad \int_0^{2\pi} |F'(re^{i\phi})|^2 d\phi = 2\pi\left[|a|^2 + \sum_{j\geq 1} j^2 |a_j|^2 r^{-2(j+1)}\right].$$

Substituting (III) into (I) and applying the inequalities of (α) to the $|a_j|$, we obtain

$$|a|^2 \geq k^2\pi^{-2}r^{-2} - \sum_{j\geq 1} j^2 r^{-2(j+1)},$$

for all $r>1$. If we set $u=r^{-2}$, $0<u<1$, we obtain

$$\text{(IV)} \qquad |a|^2 \geq \max_{0<u<1} [k^2\pi^{-2}u - u^2(1+u)(1-u)^{-3}] = [g(k)]^2.$$

Indeed, a simple calculation shows that the maximum above is positive.

For Mergelyan's proof in [2], only the special case $k=\frac{1}{4}$ is necessary. We remark, finally, that the inequality of (β) can be strengthened under the assumption that the images under F of the sets $|w|>1$ and $|w|=1$ are disjoint (an assumption which holds if F is one-to-one for $|w|>1$. In that case, we let S_1 be the complement of $F(|w|>1)$. The diameter of S_1, which we denote by k_1, will be greater than k. By translating z, we may assume that 0 lies in S_1, with our only expense a change in b. Then S_1 lies in the set $|z|\leq k_1$, and the image under F of $|w|=1$ will lie in S_1. The inequalities (α) on $|a_j|$ become $|a_j|\leq k_1$, and proceeding as above, we obtain instead of (IV),

$$\text{(V)} \qquad |a|^2 \geq |k_1|^2[g(1)]^2 \geq k^2[g(1)]^2.$$

Hence $|a|/k$ is bounded from below.

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A NOTE ON THE EXTENSION OF LINEAR FUNCTIONALS

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Bourbaki [1] has suggested a method of deducing the Hahn-Banach extension theorem from its "geometrical form" (essentially Mazur's theorem on the separation of convex sets by hyperplanes). The purpose of this note is to show that the method can be made to yield an extension theorem, similar to one re-

cently announced by Nakano [2], which is somewhat sharper than the original Hahn-Banach theorem [3].

We define a *convex functional*, on a real vector space $\mathbf{X}$, to be a real-valued function g such that

$$g(\lambda\mathbf{x} + (1-\lambda)\mathbf{x}') \leqq \lambda g(\mathbf{x}) + (1-\lambda)g(\mathbf{x}'),$$

whenever $\mathbf{x}$, $\mathbf{x}'$ belong to $\mathbf{X}$ and $0 \leqq \lambda \leqq 1$. For this condition to be satisfied, it is evidently sufficient, but not necessary, that

$$g(\mathbf{x}+\mathbf{x}') \leqq g(\mathbf{x}) + g(\mathbf{x}') \quad \text{and} \quad g(\rho\mathbf{x}) = \rho g(\mathbf{x}),$$

for all $\mathbf{x}$, $\mathbf{x}'$ in $\mathbf{X}$ and all $\rho \geqq 0$. Thus the following theorem includes the Hahn-Banach theorem as a special case.

Let g be a convex functional on a real vector space $\mathbf{X}$, let $\mathbf{X}_0$ be a vector subspace of $\mathbf{X}$, and let f_0 be a linear functional on $\mathbf{X}_0$ such that $f_0(\mathbf{x}) \leqq g(\mathbf{x})$ for all $\mathbf{x}$ in $\mathbf{X}_0$. Then there is a linear functional f on $\mathbf{X}$ such that $f(\mathbf{x}) = f_0(\mathbf{x})$ for all $\mathbf{x}$ in $\mathbf{X}_0$, and $f(\mathbf{x}) \leqq g(\mathbf{x})$ for all $\mathbf{x}$ in $\mathbf{X}$.

To prove this theorem, let $\mathbf{X}^\natural$ be the direct sum of $\mathbf{X}$ and the field of real numbers. Let C be the set of all points $(\mathbf{x}, \xi)$ of $\mathbf{X}^\natural$ which lie above the graph of g, in the sense that $g(\mathbf{x}) < \xi$. The convexity of g ensures that C is a convex set. Let $(\mathbf{x}_0, \xi_0)$ be any point of C, and $(\mathbf{x}, \xi)$ any point of $\mathbf{X}^\natural$. Then, if $0 \leqq \lambda \leqq 1$,

$$\begin{aligned} g(\mathbf{x}_0 + \lambda\mathbf{x}) - (\xi_0 + \lambda\xi) &= g(\lambda(\mathbf{x}_0 + \mathbf{x}) + (1-\lambda)\mathbf{x}_0) - \xi_0 - \lambda\xi \\ &\leqq \lambda g(\mathbf{x}_0 + \mathbf{x}) + (1-\lambda)g(\mathbf{x}_0) - \xi_0 - \lambda\xi \\ &= \lambda\{g(\mathbf{x}_0 + \mathbf{x}) - \xi_0 - \xi\} - (1-\lambda)\{\xi_0 - g(\mathbf{x}_0)\}. \end{aligned}$$

Since $\xi_0 - g(\mathbf{x}_0) > 0$, it follows that, when λ is a sufficiently small positive number, $g(\mathbf{x}_0+\lambda\mathbf{x}) - (\xi_0+\lambda\xi) < 0$, so that the point $(\mathbf{x}_0, \xi_0)+\lambda(\mathbf{x}, \xi)$ belongs to C. This property, together with the fact that C is convex, ensures that C is an open set in the largest topology with which $\mathbf{X}^\natural$ is a locally convex space.

Now let S be the graph of f_0, consisting of all points $(\mathbf{x}, f_0(\mathbf{x}))$ with $\mathbf{x}$ in $\mathbf{X}_0$. Then S does not meet C, and is a subspace of $\mathbf{X}^\natural$: hence, by the separation theorem, there is a linear functional F on $\mathbf{X}^\natural$ such that $F(\mathbf{x}, \xi) > 0$ when $(\mathbf{x}, \xi)$ is in C and $F(\mathbf{x}, \xi) = 0$ when $(\mathbf{x}, \xi)$ is in S. Since $(0, \xi)$ is in C when ξ is large, $F(0, 1) > 0$. Writing

$$f(\mathbf{x}) = \xi - F(\mathbf{x}, \xi)/F(0, 1),$$

we define on $\mathbf{X}$ a linear functional f, independent of ξ. When $\mathbf{x}$ is in $\mathbf{X}_0$, we can take $\xi = f_0(\mathbf{x})$, showing that $f(\mathbf{x}) = f_0(\mathbf{x})$. Also, for any $\mathbf{x}$ in $\mathbf{X}$, $f(\mathbf{x}) < \xi$ if $g(\mathbf{x}) < \xi$, so that f is dominated by g, as required.

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ON THE DERIVATIVE OF AN ANALYTIC FUNCTION

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Since the days of Riemann it has been the custom to define an analytic function $f(z)=u(x, y)+iv(x, y)$ by the requirement of differentiability, or, what amounts to the same thing, the requirement that the functions u and v possess total differentials and satisfy the Cauchy-Riemann equations

$$u_x = v_y, \qquad u_y = -v_x. \tag{1}$$

If these conditions hold in a neighborhood $|z-z_0|<\epsilon$ then, as shown in the textbooks on the subject, the four derivatives are necessarily continuous and $f'(z)=u_x+iv_x$ is again an analytic function in the neighborhood.

For people who like their mathematics neat it has always been a source of minor annoyance that the continuity and differentiability of $f'(z)$ could not be proved without recourse to complex integration. It apparently is felt that this procedure violates the time-honored division of the calculus into its differential and integral varieties.

In a recently published paper [1], R. L. Plunkett has succeeded in showing that the continuity of $f'(z)$ can be proved—with the help of a rather heavy dose of modern topological analysis—without even mentioning the notion of integration. The aim of the present note is much less ambitious. All the author wants to do is indicate a proof of the continuity and differentiability of $f'(z)$ which uses only ordinary Lebesgue integration and is formally very simple. The proof will be carried out under the simplifying assumption that $|f'(z)|<M<\infty$ in the given neighborhood. Purists may object that complex integration or, at any rate, Green's theorem is lurking in the background, barely visible but still there. Far from desiring to argue this point, the author is content to offer the following proof as a mere pedagogic device.

It is clearly sufficient to carry out the proof under the assumption that the conditions in question hold in the closed disk $|z|\leqq 1$. By a well-known theorem (see, e.g. [3], p. 368), the relation

$$\int_a^b g'(x)dx = g(b) - g(a)$$

holds if $g'(x)$ exists and is uniformly bounded in $[a, b]$. Hence

$$u(1, \theta) - u(0) = \int_0^1 u_r(r, \theta)dr, \qquad z = re^{i\theta}.$$

Both sides are continuous in θ, and thus

$$\int_0^{2\pi} u(1, \theta)d\theta - 2\pi u(0) = \int_0^{2\pi}\int_0^1 u_r(r, \theta)drd\theta. \tag{2}$$

We choose a small positive constant ϵ and remark that in the region $\epsilon\leqq r\leqq 1$,

$0 \leqq \theta \leqq 2\pi$, $u(r, \theta)$ is continuous and $u_r(r, \theta)$ exists. It follows from another well-known result (see, e.g. [2], p. 170) that u_r is measurable over this region. Since $|u_r| \leqq M$, it is also summable and Fubini's theorem may be applied. Hence

$$\int_0^{2\pi} \int_\epsilon^1 u_r(r, \theta) dr d\theta = \int_\epsilon^1 \int_0^{2\pi} u_r(r, \theta) d\theta dr.$$

By (1), $ru_r = v_\theta$, and we have

$$r \int_0^{2\pi} u_r(r, \theta) d\theta = \int_0^{2\pi} v_\theta(r, \theta) d\theta = v(r, 2\pi) - v(r, 0) = 0,$$

since v_θ exists and is uniformly bounded for $\theta \in [0, 2\pi]$. (2) thus leads to

$$\left| \int_0^{2\pi} u(1, \theta) d\theta - 2\pi u(0) \right| = \left| \int_0^{2\pi} \int_0^\epsilon u_r(r, \theta) dr d\theta \right| \leqq 2\pi \epsilon M,$$

which shows that $u(r, \theta)$ is subject to the mean-value theorem

$$u(0) = \frac{1}{2\pi} \int_0^{2\pi} u(1, \theta) d\theta.$$

The same procedure yields the corresponding result for $v(r, \theta)$, and we thus obtain

$$f(0) = \frac{1}{2\pi} \int_0^{2\pi} f(e^{i\theta}) d\theta = \frac{1}{2\pi i} \int_{|z|=1} f(z) \frac{dz}{z}, \qquad z = e^{i\theta}. \tag{3}$$

The transformation $z = (\zeta + t)(1 + \bar{\zeta} t)^{-1} (|\zeta| < 1)$ maps $|z| \leqq 1$ onto $|t| \leqq 1$. Since the function

$$g(t) = f\left(\frac{\zeta + t}{1 + \bar{\zeta} t} \right)$$

is differentiable in $|t| \leqq 1$ if the same is true of $f(z)$ in $|z| \leqq 1$, we may apply (3). Hence,

$$f(\zeta) = g(0) = \frac{1}{2\pi i} \int_{|t|=1} g(t) \frac{dt}{t} = \frac{1}{2\pi i} \int_{|t|=1} f(z) \frac{dt}{t}.$$

But $t = (z - \zeta)(1 - \bar{\zeta} z)^{-1}$, i.e.,

$$\frac{dt}{t} = \left[\frac{1}{z - \zeta} + \frac{\bar{\zeta}}{1 - \bar{\zeta} z} \right] dz,$$

and we have

$$f(\zeta) = \frac{1}{2\pi i} \int_{|z|=1} \frac{f(z)}{z - \zeta} dz + \frac{\bar{\zeta}}{2\pi i} \int_{|z|=1} \frac{f(z)}{1 - \bar{\zeta} z} dz. \tag{4}$$

If $|\zeta| = |\xi + i\eta| < 1$, differentiation with respect to ξ and η under the integral sign is permissible for trivial reasons. If we apply to (4) the operation $\frac{1}{2}(\partial/\partial\xi - i\partial/\partial\eta)$, the second integral drops out and we obtain the identity

$$f'(\zeta) = \frac{1}{2\pi i}\int_{|z|=1} \frac{f(z)}{(z-\zeta)^2}\,dz,$$

which contains the desired result.

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A CLASS OF PARTIALLY ORDERED SETS

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In this note we consider a class of partially ordered sets, which we call *fairly well-ordered*, and show that they have some simple, desirable properties. A linearly ordered set is fairly well-ordered if and only if it is well-ordered. Moreover, we show in Proposition 1 that any subset, finite cartesian product, or image under an order-preserving map of fairly well-ordered sets is fairly well-ordered.

Fairly well-ordered sets are characterized by any of the four properties which the following theorem proves equivalent. The most useful consequence of our results is probably the fact—no doubt already known to many—that the product of finitely many copies of the positive integers has property (d) below. This result is used by J. Ceder to prove a topological theorem in [1], and it was a conjecture by Ceder which provided the original motivation for this study.

The author is grateful for some very helpful conversations with J. Isbell, to whom the present form of this paper can largely be credited.

THEOREM 1. *The following properties of a partially ordered set X are equivalent.*

(a) *Every sequence $\{x_i\}_{i=1}^{\infty}$ in X has a subsequence $\{x_{i_n}\}_{n=1}^{\infty}$ such that $x_{i_{n+1}} \geqq x_{i_n}$ for all n.*

(b) *Every infinite subset of X contains an infinite ascending chain (i.e., a sequence $\{x_n\}_{n=1}^{\infty}$ such that $x_{n+1} > x_n$ for all n).*

(c) *X contains no infinite descending chain* (i.e. *no sequence $\{x_n\}_{n=1}^{\infty}$ with $x_{n+1} < x_n$ for all n), and no infinite, pairwise incomparable subset.*

(d) *Every $S \subset X$ has a finite subset $M(S)$ such that each $x \in S$ is $\geqq$ some $x' \in M(S)$.*

Proof. The implications $(a) \to (b) \to (c) \to (d)$ are trivial, and can be left to the reader to check. It remains to prove $(d) \to (a)$, for which we need the following lemma, known as König's infinity lemma:

LEMMA 1. *Let* $\{E_n\}_{n=1}^{\infty}$ *be a sequence of nonempty, pairwise disjoint, finite sets, and let* $f_n: E_{n+1} \to E_n$ *be a function for each* n. *Then we can select from each* E_n *an element* e_n *such that* $f_n(e_{n+1}) = e_n$ *for all* n.

Proof. Call $e \in E_n$ a successor of $e' \in E_m$ if $n > m$ and $[f_{n-1} \circ f_{n-2} \circ \cdots \circ f_m](e) = e'$. Since E_1 is finite and $\bigcup_{n=2}^{\infty} E_n$ infinite, there exists an $e_1 \in E_1$ with infinitely many successors. Since e_1 has finitely many successors in E_2, it follows that one of these, say e_2, has infinitely many successors. Continuing in this fashion, we inductively construct the desired sequence $\{e_n\}_{n=1}^{\infty}$, and that proves the lemma.

We now proceed with the proof that $(d) \to (a)$ in Theorem 1. By induction we shall define nonempty, finite sets I_n $(n = 1, 2, \cdots)$ of positive integers as follows: Pick I_1 such that $\{x_i\}_{i \in I_1} = M(\{x_i\}_{i=1}^{\infty})$. Having defined $I_1, \cdots, I_n$, let $a_n = \max(\bigcup_{i=1}^{n} I_i)$, and then pick I_{n+1} from among $\{k \mid k > a_n\}$ such that

$$\{x_i\}_{i \in I_{n+1}} = M(\{x_k \mid k > a_n\}).$$

It follows that for each n there is a function $f_n: I_{n+1} \to I_n$ such that $x_{f_n(i)} \leqq x_i$ for all $i \in I_{n+1}$.

By Lemma 1, we can pick from each I_n an element i_n such that $i_n = f(i_{n+1})$ for all n. But then $\{x_{i_n}\}_{n=1}^{\infty}$ is the desired subsequence of $\{x_i\}_{i=1}^{\infty}$, and the proof of the theorem is complete.

DEFINITION 1. *A partially ordered set is* fairly well-ordered *if it satisfies any of the four equivalent conditions in Theorem* 1 *above.*

PROPOSITION 1.

(1) *A linearly ordered set is fairly well-ordered if and only if it is well-ordered.*

(2) *A subset of a fairly well-ordered set is fairly well-ordered.*

(3) *The image of a fairly well-ordered set, under an order-preserving map* f (i.e. $x \leqq y$ *implies* $f(x) \leqq f(y)$), *is fairly well-ordered.*

(4) *The cartesian product of finitely many fairly well-ordered sets (with the natural order:* $(x_1, \cdots, x_n) \leqq (y_1, \cdots, y_n)$ *means* $x_1 \leqq y_1, \cdots, x_n \leqq y_n$), *is fairly well-ordered.*

Proof. Assertions (1) and (2) follow from any of the characterizations in Theorem 1; (3) follows from (a) or (d), while (4) follows from (a).

Only the last statement requires proof. First, observe that it suffices to prove (4) for the product of *two* spaces. So suppose that X and Y are fairly well-ordered, and let $\{(x_i, y_i)\}_{i=1}^{\infty}$ be a sequence in $X \times Y$. By (a), $\{x_i\}_{i=1}^{\infty}$ has a subsequence $\{x_{i_n}\}_{n=1}^{\infty}$ with $x_{i_{n+1}} \geqq x_{i_n}$ for all n. Again by (a), $\{y_{i_n}\}_{n=1}^{\infty}$ has a subsequence $\{y_{i_{n_p}}\}_{p=1}^{\infty}$ such that $\{y_{i_{n_{p+1}}}\} \geqq y_{i_{n_p}}$ for all p. It follows that $\{(x_{i_{n_p}}, y_{i_{n_p}})\}_{p=1}^{\infty}$ is the desired subsequence of $\{(x_i, y_i)\}_{i=1}^{\infty}$, and that completes the proof.

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ON THE SYMMETRY OF SEMISIMPLE RINGS

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By a semisimple ring we mean a ring containing no nonzero nilpotent left ideal and satisfying the descending chain condition for left ideals. It is known that for semisimple rings a complete symmetry prevails between concepts involving "right-hand" notions and between their "left-hand" analogues. Thus to any characterization of these rings there exists a "dual" characterization which arises if for the "right-hand" and "left-hand" concepts occurring in the original characterization we substitute the corresponding "left-hand" and "right-hand" ones. For example, by the well-known theorem of Noether an arbitrary ring R is semisimple if and only if it has a right unit element and can be decomposed into the direct sum of minimal left ideals (see [2], or [3], Sec. 123), and by the above-mentioned duality we obtain that the ring R is semisimple if and only if R has a left unit element and can be decomposed into the direct sum of minimal right ideals.* To the best of our knowledge the demonstration of the equivalence of the two dual characterizations of semisimple rings has been effected so far only in an indirect way, mostly with the aid of the self-dual Wedderburn-Artin theorem. It is the purpose of this note to give a simple and straightforward proof for the equivalence of the two dual noetherian characterizations.

We are going to prove the following theorem:

If a ring R has a right unit element and admits a decomposition into a direct sum of a finite number of minimal left ideals, then R has also a left unit element and can be decomposed into a direct sum of finitely many minimal right ideals, and conversely.

It will clearly be sufficient to prove the first assertion of the theorem.

Let e be a right unit element of the ring R, and let the direct decomposition

$$R = L_1 + \cdots + L_m \tag{1}$$

hold, where the L_i $(i=1, \cdots, m)$ are mininal left ideals of R. In view of (1) we have for e the decomposition

$$e = e_1 + \cdots + e_m \qquad (e_i \in L_i;\ i = 1, \cdots, m). \tag{2}$$

First we show that $L_i = Re_i$ $(i=1, \cdots, m)$ and $e_1, \cdots, e_m$ are orthogonal idempotents.† In view of (2) we have for each component of e, $e_i = e_i e = e_i e_1 + \cdots$

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* In E. Noether's original formulation of the theorem there occurs a "two sided unit element." It was one of us who remarked that it is sufficient to postulate the existence of a one sided unit element [1].

† The proof of this assertion is essentially the same as that of Lemma 5 in §123 of [3]. We give here this proof only for completeness sake and in view of the circumstance that in our considerations we use only a right unit element instead of a two sided one.

$+e_i^2+\cdots+e_ie_m$, whence by the uniqueness of the decomposition the relations

$$e_ie_k=\begin{cases}e_i & \text{for } i=k,\\ 0 & \text{for } i\neq k,\end{cases} \tag{3}$$

follow. Since by $e_i^2=e_i(\neq 0)$ we have the relation $Re_i\neq 0$ and by $e_i\in L_i$ the relation $Re_i\subseteq L_i$, the minimality of L_i implies $Re_i=L_i$.

Let us now show that e is a left unit element in R. Consider an arbitrary element r of R, which, by (2), admits the decomposition

$$r=re=re_1+\cdots+re_m. \tag{4}$$

For a fixed i $(1\leqq i\leqq m)$ the set of all elements of the form

$$ere_i-re_i \tag{5}$$

is a left ideal $L(\subseteq Re_i)$ of R. By the minimality of Re_i we must have either $L=Re_i$ or $L=0$. The case $L=Re_i$ is, however, impossible since the product of two elements of (5) is always zero, and so L cannot contain the idempotent element e_i. Now $L=0$ implies

$$ere_i=re_i, \qquad (i=1,\cdots,m). \tag{6}$$

Hence, by (4), $er=ere_1+\cdots+ere_m=re_1+\cdots+re_m=re=r$ and this proves e to be a left unit element. Accordingly we have, by (2), for any element $r(\in R)$,

$$r=e_1r+\cdots+e_mr. \tag{7}$$

On the other hand, since from the relation $e_1r_1+\cdots+e_mr_m=0$ $(r_1,\cdots,r_m\in R)$ it follows by left multiplication with e_i $(i=1,\cdots,m)$ in view of (3) that $e_ir_i=0$, and hence any element of the ring R admits a unique representation as a sum of elements belonging to the right ideals $e_1R,\cdots,e_mR$, respectively. Consequently we have the direct decomposition

$$R=e_1R+\cdots+e_mR. \tag{8}$$

In order to prove the minimality of the right ideals e_iR $(i=1,\cdots,m)$, it will be sufficient to show that for an arbitrary fixed element $e_is(\neq 0;\ s\in R)$ there exists a $t(\in R)$ such that

$$e_is\cdot t=e_i. \tag{9}$$

First of all we remark that, by (4), the element $e_is(\neq 0)$ can be written in the form $e_is=e_ise_1+\cdots+e_ise_m$, where we may consider the component e_ise_k to be different from zero. Let us now consider the left ideal $Re_i\cdot e_ise_k$. By the idempotency of e_i and by the minimality of Re_k we have for this ideal,

$$Re_i\cdot e_ise_k=Re_k. \tag{10}$$

From (10) there follows $e_kRe_i\cdot e_ise_k=e_kRe_k$, and this assures the existence of an element $s^*(\in R)$ for which

$$e_k s^* e_i \cdot e_i s e_k = e_k \tag{11}$$

holds. Now we are going to prove that

$$e_i s e_k \cdot e_k s^* e_i = e_i. \tag{12}$$

In order to shorten our notations, we put $e_i s e_k \cdot e_k s^* e_i = z$. Multiplying equation (11) from the left by $e_i s e_k$ we obtain with the aid of $e_i s e_k \neq 0$ the relation $z \neq 0$, and again by (11) the equality

$$z^2 = z. \tag{13}$$

Finally, by the definition of z and by (3) we have

$$e_i z = z. \tag{14}$$

On the other hand, by $Rz \neq 0$, $Rz \subseteq Re_i$ and by the minimality of Re_i we have $Rz = Re_i$. From this $yz = e_i$ follows for a suitable element $y (\in R)$. However, by (13) this implies

$$e_i z = yz^2 = yz = e_i, \tag{15}$$

and (14) and (15) together prove (12). Since by (12) the equation (9) has the solution $t = e_k s^* e_i$, the right ideals $e_i R$ $(i = 1, \cdots, m)$ are minimal right ideals in R, and this completes the proof of the theorem.

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REMARKS ON THE EULER-MACLAURIN AND BOOLE SUMMATION FORMULAS

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This note contains a few simple observations which provide a point of view from which it is especially easy to see the structural similarity of the Euler-Maclaurin and Boole formulas and the analogy which each bears to Taylor's formula.

For any function f which is continuous on the closed interval $[x, x+1]$ let $If(x)$ denote the integral average $\int_0^1 f(x+t)dt$ and let $Af(x)$ denote the arithmetic average $\frac{1}{2}[f(x)+f(x+1)]$.

One sees immediately that for every polynomial ϕ there exists a unique polynomial ϕ^* such that $I\phi^* = \phi$ and a unique polynomial ϕ^{**} such that $A\phi^{**} = \phi$. (One sees also that ϕ^* and ϕ^{**} have the same degree as ϕ.)

We now define the Bernouilli polynomial B_n of degree n to be the unique polynomial solution of the equation $IB_n(x) = x^n$, and the Euler polynomial E_n of degree n to be the unique polynomial solution of the equation $AE_n(x) = x^n$.

(For the Euler polynomials the definition is classical. For the Bernouilli polynomials, for which the classical definition is stated in terms of the equation $\Delta B_n(x) = nx^{n-1}$, the present definition has the advantage that the troublesome arbitrary constant in the polynomial solution of $\Delta y = nx^{n-1}$ does not appear, even transiently. In conjunction with the present definition of B_n, the uniqueness of the polynomial solution of the equation $I\phi^* = \phi$ can be used to simplify the derivations of the well-known properties of the Bernouilli polynomials. For example, the formula $B_n(x+h) = \sum_{s=0}^{n} \binom{n}{s} h^s B_{n-s}(x)$ follows at once from the fact that if $g_1(x)$ and $g_2(x)$ are the right and left members of the formula in question, then we see immediately that $Ig_1(x)$ and $Ig_2(x)$ are both equal to $(x+h)^n$, and therefore $g_1 = g_2$.)

The Euler-Maclaurin and Boole formulas can now be written

$$f(x) = \sum_{k=0}^{n} \frac{If^{(k)}(a)}{k!} B_k(x - a) + \text{remainder}, \tag{1}$$

$$f(x) = \sum_{k=0}^{n} \frac{Af^{(k)}(a)}{k!} E_k(x - a) + \text{remainder}, \tag{2}$$

respectively, to be compared with each other and with Taylor's formula

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!} (x - a)^k + \text{remainder}. \tag{3}$$

The summations appearing in the right members of (1), (2), and (3) are characterized as the uniquely determined polynomials $p_1(x)$, $p_2(x)$, and $p_3(x)$ of degree n such that $Ip_1^{(k)}(a) = If^{(k)}(a)$, $Ap_2^{(k)}(a) = Af^{(k)}(a)$, and $p_3^{(k)}(a) = f^{(k)}(a)$, $(k = 0, 1, \cdots, n)$, respectively.

To find a formula for the remainder $R_n(x)$ in (1) we may proceed as follows: Let $h = x - a$, and assume that $0 \leqq h \leqq 1$. Then

$$R_0(x) = f(a + h) - If(a)B_0(h) = \int_0^1 [f(a + h) - f(a + t)]dt$$

$$= \int_0^1 \int_t^h f'(a + s)dsdt.$$

If the order of integration is reversed, the double integral can be rewritten as $\int_0^1 V(s, h)f'(a+s)ds$, where $V(s, h) = s$ $(0 < s < h)$, $s - 1$ $(h < s < 1)$. Hence

$$R_1(x) = \int_0^1 V(s, h)f'(a + s)ds - \int_0^1 f'(a + s)ds(h - \tfrac{1}{2}) = \int_0^1 W(s, h)f'(a + s)ds,$$

where $W(s, h) = s - h + \frac{1}{2}$ $(0 < s < h)$, $s - h - \frac{1}{2}$ $(h < s < 1)$. Thus $R_1(x) = -\int_0^1 \overline{B}_1(h - s)f'(a + s)ds$, where $\overline{B}_1(x) = B_1(x)$ $(0 \leqq x < 1)$ and $\overline{B}_1(x + 1) = \overline{B}_1(x)$ (all x). From here on the usual remainder formula

$$R_n(x) = -\int_0^1 \overline{B}_n(h-s)\frac{f^{(n)}(a+s)}{n!}\,ds$$

can be obtained by repeated integrations by parts, just as in the case of Taylor's formula with integral remainder. A similar discussion applies to the remainder term in (2).

In brief, the essential step in passing from Taylor's formula to the Euler-Maclaurin formula, or to the Boole formula, is the replacement of each derivative at a by the *integral average* of the derivative over $[a, a+1]$, or by the *arithmetic average* of the values of the derivative at the points a and $a+1$.

CLASSROOM NOTES

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THE EFFECTS OF SINGULAR COLLINEATIONS

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Generally in an elementary study of collineations, considerable attention is given to the classification of the nonsingular ones as to the nature of their fixed points, the conditions for their periodicity, etc. Also it is usually pointed out that when a collineation is singular there is a certain subspace of points which have no correspondent (since all their homogeneous coordinates become zero), while the remaining points of the space undergoing transformation are carried into another fixed subspace. For example, one often finds the statement that if a collineation in the plane ($\rho x_i' = \sum_{j=1}^{3} a_{ij}x_j$) has a matrix of rank 2, there is one point which has no correspondent, while the remaining points of the plane all have their correspondents on a fixed line. Seldom is anything said about a singular collineation having a matrix whose order exceeds its rank by more than one, and, though an exhaustive search has not been made, nowhere has any mention been found of the manner in which this "piling up" of points on a subspace is effected. The purpose of this note is to explain how this "piling up" is effected in the general case where an n-space, S_n, is subjected to a transformation whose matrix has rank $n-k$.

Before considering the most general case, perhaps we should restate what appears well known concerning a collineation in S_n whose matrix has rank n. Here it is easily shown that there is one singular point F, or an S_0, which has no correspondent, and that all other points of S_n are carried into a fixed S_{n-1}. The collineation then induces a collineation in the fixed S_{n-1}, and this collinea-

tion in S_{n-1} may be of any type, including the identity and even singular ones. We shall now prove

THEOREM 1. *All points of the line joining a point B to the singular point F go into one and the same point on the fixed* S_{n-1}.

The collineation T of rank m may be defined by the equations

$$\rho x_i' = a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{i,n+1}x_{n+1}, \qquad i = 1, \cdots, n;$$
$$\rho x_{n+1}' = 0.$$

Let the coordinates of the singular point F be $(f_1, f_2, \cdots, f_{n+1})$ and let $B(b_1, b_2, \cdots, b_{n+1})$ be any other point in S_n. Then *any* point of FB has coordinates

$$x_i = \lambda f_i + \mu b_i, \qquad i = 1, \cdots, n+1,$$

and by T the coordinates become

$$\begin{aligned}\rho x_i' &= a_{i1}(\lambda f_1 + \mu b_1) + a_{i2}(\lambda f_2 + \mu b_2) + \cdots + a_{i,n+1}(\lambda f_{n+1} + \mu b_{n+1}) \\ &= \lambda(a_{i1}f_1 + a_{i2}f_2 + \cdots + a_{i,n+1}f_{n+1}) + \mu(a_{i1}b_1 + a_{i2}b_2 + \cdots + a_{i,n+1}b_{n+1}) \\ &= \mu \sum_{j=1}^{n+1} a_{ij}b_j, \text{ since } F \text{ is the singular point.}\end{aligned}$$

Thus this point is precisely the point into which B is carried by T.

We easily generalize these remarks to observe that if the rank of the matrix of the transformation T is $n-k$, there is a singular S_k, and all other points of S_n are carried into a fixed S_{n-k-1}. The transformation induces a collineation in this fixed S_{n-k-1}, and, generalizing Theorem 1, we state

THEOREM 2. *All points of the* S_{k+1} *determined by the singular space* S_k *and a point B, not in* S_k, *are carried by the transformation T into one and the same point of the fixed* S_{n-k-1}, *the correspondent of the point B.*

The collineation T of rank $n-k$ may be written in the form

$$\rho x_i' = a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{i,n+1}x_{n+1}, \qquad i = 1, \cdots, n-k;$$
$$\rho x_j' = 0, \qquad j = (n-k+1), \cdots, (n+1).$$

Let the singular S_k, defined by equating to zero the $n-k$ independent linear forms above, contain the $k+1$ independent points $F_1, F_2, \cdots, F_{k+1}$, where in this set the point F_i has coordinates $(f_{i1}, f_{i2}, \cdots, f_{i,n+1})$. These points also suffice to determine the singular S_k. Let $B(b_1, \cdots, b_{n+1})$ be any point of S_n not in the singular S_k. Then the points F_i and B determine an S_{k+1}, any point of which has the coordinates

$$x_i = \lambda_1 f_{1i} + \lambda_2 f_{2i} + \cdots + \lambda_{k+1} f_{k+1,i} + \mu b_i.$$

Applying the collineation T, we have

$$\rho x_i' = a_{i1}\left[\mu b_1 + \sum_{i=1}^{k+1} \lambda_i f_{i1}\right] + a_{i2}\left[\mu b_2 + \sum_{i=1}^{k+1} \lambda_i f_{i2}\right] + \cdots$$

$$+ a_{i,n+1}\left[\mu b_{n+1} + \sum_{i=1}^{k+1} \lambda_i f_{i,n+1}\right]$$

$$= \mu[a_{i1}b_1 + a_{i2}b_2 + \cdots + a_{i,n+1}b_{n+1}] + \lambda_1 \sum_{j=1}^{n+1} a_{ij}f_{ij} + \cdots + \lambda_{k+1} \sum_{j=1}^{n+1} a_{ij}f_{k+1,j}.$$

But since each of the points F_i is a point of the singular S_k, each of the summations above is zero, and we get

$$\rho x_i' = \mu \sum_{j=1}^{n+1} a_{ij}b_j;$$

thus we have shown that all points of the S_{k+1} which are outside the singular S_k are carried into one and the same point, the correspondent of the point B.

AN EXAMPLE IN DIFFERENTIAL EQUATIONS: THE N-BODY PROBLEM

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It is easy for a student in an elementary course in differential equations to acquire three inaccurate conceptions concerning the power series method: (i) that the method is suitable for a single equation but not for systems; (ii) that the method is suitable for linear equations only; (iii) that a solution by this method is not satisfactory unless the coefficients of the series are given explicitly as a function of n.

Misconception (iii) can be dispelled by pointing out that a recursive definition of the coefficients is more useful for computing than an explicit definition and can often be used directly in the ratio test for convergence of the series. Misconceptions (i) and (ii) can be dispelled by an example such as the one to follow, in which power series are obtained as (local) solutions of the n-body problem. The method follows [1] and [2] and has been programmed by Norman Levine for the IBM 704 computer.

Let there be given n particles distributed in space, the mass and position vector of the ith particle being denoted by M_i and $\mathbf{X}_i = (X_{i1}, X_{i2}, X_{i3})$. In the sequel, the letters i, j, k, n, and p denote integers which satisfy (unless the contrary be stated) $1 \leqq i \leqq n$, $1 \leqq j \leqq n$, $j \neq i$, $1 \leqq h \leqq 3$, $0 \leqq k < \infty$, and $1 \leqq p \leqq 3$. Now define

$$R_{ij} = \sum_h (X_{ih} - X_{jh})^2, \qquad S_{ij} = (R_{ij})^{-3/2}. \tag{1}$$

By Newton's Law of Gravitation, the force exerted by M_j on M_i has magnitude GM_iM_j/R_{ij} and is directed along the unit vector $R_{ij}^{-1/2}(\mathbf{X}_j - \mathbf{X}_i)$. Thus the total force on M_i is

$$\sum_j GM_iM_j(\mathbf{X}_j - \mathbf{X}_i) \div R_{ij}R_{ij}^{1/2}.$$

By Newton's 2nd Law of Motion, then

$$\frac{d^2}{dt^2}\mathbf{X}_i = \sum_j GM_jS_{ij}(\mathbf{X}_j - \mathbf{X}_i). \tag{2}$$

By differentiation of the second of equations (1), we have

$$2R_{ij}\frac{d}{dt}S_{ij} + 3S_{ij}\frac{d}{dt}R_{ij} = 0. \tag{3}$$

Power series are next introduced:

$$X_{ih} = \sum_k X_{ihk}t^k, \qquad R_{ij} = \sum_k R_{ijk}t^k, \qquad S_{ij} = \sum_k S_{ijk}t^k. \tag{4}$$

Now let the series of (4) be substituted in equations (1), (3), and (2), the resulting products of series being formed by the Cauchy product formula. The identification of coefficients of like powers of t in the usual way leads to three recurrence relations:

$$R_{ijk} = \sum_h \sum_{v=0}^{k} [X_{ihv}X_{ih(k-v)} - 2X_{ihv}X_{jh(k-v)} + X_{jhv}X_{jh(k-v)}]. \tag{5}$$

$$S_{ijk} = -(2kR_{ij0})^{-1}\left\{\sum_{v=1}^{k-1} 2vR_{ij(k-v)}S_{ijv} + \sum_{v=1}^{k} 3vS_{ij(k-v)}R_{ijv}\right\}. \tag{6}$$

$$X_{ih(k+1)} = (k^2 + k)^{-1}\sum_{v=0}^{k-1}\sum_j GM_jS_{ij(k-v-1)}(X_{jhv} - X_{ihv}). \tag{7}$$

In (6) and (7), $0 < k < \infty$. In (6) the first sum is taken 0 when $k = 1$.

If initial conditions of position and velocity are known for all n bodies, then the numbers X_{ih0} and X_{ih1} are prescribed. From these one obtains R_{ij0} via (5); also $S_{ij0} = R_{ij0}^{-3/2}$. Thereafter, formulas (5), (6), and (7) may be employed cyclically to determine the remaining coefficients. A lower bound for the radius of convergence may be given by the Cauchy-Poincaré Theorem. This is done, for example, in [3], pp. 23–25. This lower bound need not be a good approximation to the radius of convergence, however. In one application of the above method, the radius of convergence turned out to be approximately 10^5 times the lower bound given by the Cauchy-Poincaré Theorem.

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TRIANGLES FORMED BY LINE SEGMENTS PARALLEL TO THE ALTITUDES AND ANGLE BISECTORS OF A TRIANGLE

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The theorem that three line segments respectively parallel to the medians of a triangle, and of proportional lengths, form a triangle, and its proof by vector methods, are well known. Corresponding theorems can be stated for the angle bisectors and the altitudes, and easily proved by vector methods, as follows:

THEOREM 1. *Three line segments respectively parallel to the angle bisectors of a triangle, and of lengths proportional to the cosines of half the bisected angles, form a triangle.*

THEOREM 2. *Three line segments respectively parallel to the altitudes of a triangle, and of lengths proportional to the sines of the originating angles, form a triangle.*

To prove these theorems, let $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ be the sides of the given triangle expressed as vectors, their directions taken counter-clockwise around the triangle; $\mathbf{P}$, $\mathbf{Q}$, $\mathbf{R}$, unit vectors perpendicular to $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, respectively, their directions taken outward from the triangle; and α, β, γ, the angles opposite $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, respectively. The vectors

$$\frac{\mathbf{B}}{|\mathbf{B}|}-\frac{\mathbf{C}}{|\mathbf{C}|}, \qquad \frac{\mathbf{C}}{|\mathbf{C}|}-\frac{\mathbf{A}}{|\mathbf{A}|}, \qquad \frac{\mathbf{A}}{|\mathbf{A}|}-\frac{\mathbf{B}}{|\mathbf{B}|}$$

are respectively parallel to the angle bisectors and of lengths

$$2\cos\tfrac{1}{2}\alpha, \qquad 2\cos\tfrac{1}{2}\beta, \qquad 2\cos\tfrac{1}{2}\gamma.$$

Since their sum is zero, the first theorem follows. Also,

$$\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = k\mathbf{P}\sin\alpha, \qquad \mathbf{B}\times(\mathbf{C}\times\mathbf{A}) = k\mathbf{Q}\sin\beta,$$

and $\mathbf{C}\times(\mathbf{A}\times\mathbf{B})=k\mathbf{R}\sin\gamma$, where $k=|\mathbf{A}|\cdot|\mathbf{B}|\cdot|\mathbf{C}|$. Since $\mathbf{A}\times(\mathbf{B}\times\mathbf{C})+\mathbf{B}\times(\mathbf{C}\times\mathbf{A})+\mathbf{C}\times(\mathbf{A}\times\mathbf{B})\equiv\mathbf{O}$, $\mathbf{P}\sin\alpha+\mathbf{Q}\sin\beta+\mathbf{R}\sin\gamma=\mathbf{O}$, and the second theorem follows.

A UNIFYING EQUATION IN MATHEMATICS OF FINANCE

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1. Introduction. The essential unity of many seemingly different topics discussed in textbooks on mathematics of finance has either been ignored or not been fully appreciated by textbook writers in this subject. It is the purpose of this paper to show the common origin of many of the seemingly unrelated formulas which appear in standard textbooks under a wide variety of notations which help to obscure their interrelationships. We first recall the following symbols:

$s_{\overline{n}|i}$: The accumulated value of n payments of 1, made at the end of each period for n periods at rate i per period, at the time of the last payment.

$a_{\overline{n}|i}$: The discounted value of the same series of payments one period before the time of the first payment.

$s^{-1}_{\overline{n}|i}$: The periodic payment made at the end of each period for n periods at rate i per period which will have an accumulated value of 1 at the time of the last payment.

$a^{-1}_{\overline{n}|i}$: The periodic payment made at the end of each period for n periods at rate i per period, which has a discounted value of 1.

$s^{(p)}_{\overline{1}|i}$: The accumulated value at the end of one interest conversion period of p payments of $1/p$ made at the ends of each pth part of the interest conversion period.

The usual formulas developed in the study of (a) Sinking Funds, (b) Amortization, (c) Bonds, (d) Evaluation of Mining Property, (e) Capitalized Cost, (f) Reinvestment Rates and (g) Depreciation will be identified with a single formula,

$$R = Fi' + (F - S)s^{-1}_{\overline{n}|i} = Fi' + Ws^{-1}_{\overline{n}|i}, \tag{1}$$

where R, the annual return on an investment, F, to earn a rate i', must include a term $(F-S)s^{-1}_{\overline{n}|i}$ to restore the wearing value $W=F-S$ at the end of the useful lifetime, n, by creating a sinking fund earning a rate i.

Also a generalization of (1) will be given and applied to two examples.

2(a). Sinking funds. A debtor who desires to reduce a loan F to an amount S in n years by the sinking fund method will be required to make payments of $i'F$ to his creditor and $(F-S)s^{-1}_{\overline{n}|i}$ to a sinking fund growing at a rate i. If R represents the total annual outlay of the debtor then $R=Fi'+(F-S)s^{-1}_{\overline{n}|i}$ which is (1).

2(b). Amortization of a debt. It is customary to consider the complete amortization of a debt over a period of n years but it will serve our purpose better to consider the more general problem of the partial amortization of a debt from its initial value, F, to a final unpaid balance, S, where S may or may not be zero.

The necessary equation of value can be based on the fact that the accumulated value of the debt at the end of n years may be equated to the accumulated value of the payments made and the unpaid balance at that time.

$$F(1 + i')^n = Rs_{\overline{n}|i'} + S. \tag{2}$$

Solving this equation for R gives,

$$R = \frac{F(1 + i')^n - S}{s_{\overline{n}|i'}},$$

and, replacing $(1+i')^n$ by $1+i's_{\overline{n}|i'}$

$$R = \frac{F[1 + i's_{\overline{n}|i'}] - S}{s_{\overline{n}|i'}} = Fi' + (F - S)s_{\overline{n}|i'},$$

which is (1) for $i=i'$.

2(c). Bonds. Making use of the formula which states that the purchase price F of a bond is equal to the discounted value of the redemption price, S, for n years plus the discounted value of the coupons R, both at the investor's rate i', we may write

$$F = Sv^n + R \cdot a_{\overline{n}|i'} = [F - (F - S)]v^n + Ra_{\overline{n}|i'},$$

from which follows

$$R = \frac{F(1 - v^n) + (F - S)v^n}{a_{\overline{n}|i'}} = Fi' + (F - S)s^{-1}_{\overline{n}|i'},$$

which is (1) for $i=i'$.

2(d). Evaluation of mining property. The usual derivation of the formula for finding the value of mining property yields, in the notation of this article,

$$F = \frac{Rs_{\overline{n}|i}}{1 + i's_{\overline{n}|i}} = \frac{R}{s^{-1}_{\overline{n}|i} + i'}.$$

Solving this equation for R gives $R=Fi'+Fs^{-1}_{\overline{n}|i}$ which is (1) for $S=0$.

It may be pointed out in passing that many investments may lead to negative values of S which introduce no complication whatever because the sinking fund must, in such circumstances, not only restore the original fund F but also provide enough to care for the additional obligation, S.

2(e). Capitalized cost. The capitalized cost, K, of an article costing F dollars with a scrap value S at the end of n years, is usually defined as the initial cost F plus the cost of perpetual renewals or,

$$K = F + (F - S)s^{-1}_{\overline{n}|i}/i.$$

Multiplying this equation through by i gives $Ki=R=Fi+(F-S)s^{-1}_{\overline{n}|i}$, which is identical with (1) for $i'=i$.

2(f). Reinvestment rates. A published formula* for finding the value of an enterprise, P, with final value, L, to an investor who stipulates that during n years he will earn the rate i per period on his original investment assuming that it will produce dividends D per period and that the dividends can be reinvested at rate r per period is $P=(L+Ds_{\overline{n}|r})/(1+is_{\overline{n}|r})$.

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* W. L. Hart, Mathematics of Investment (3rd ed.), Boston, 1946, (2), p. 181.

In the notation of this paper, this formula becomes

$$F = (S + rs_{\overline{n}|i})/(1 + i's_{\overline{n}|i}),$$

which on solving for R is identical with (1).

2(g). Depreciation. Among the traditional approaches to the study of depreciation there are only two which give recognition to the problem of replacement or the rate of return on a depreciating asset,—they are (i) Annuity Method and (ii) Compound Interest Method, which will be considered in this order.

(i) *Annuity method.* Following the usual development found in books on mathematics of finance, the formula for the periodic depreciation charge is derived as

$$R = [F(1 + i)^n - S]s_{\overline{n}|i}^{-1}.$$

Replacing $(1+i)^n$ by $1+is_{\overline{n}|i}$, this becomes

$$R = Fi + (F - S)s_{\overline{n}|i}^{-1},$$

an evident special case of equation (1) for $i'=i$.

(ii) *Compound interest method.* The formula* usually given in books may be written in the notation of this paper as

$$R_r = i'\left[F - \frac{W}{s_{\overline{n}|i}}\, s_{\overline{r-1}|i}\right] + \frac{W}{s_{\overline{n}|i}}(1 + i)^{r-1}, \tag{3}$$

where i' is defined as the rate of return desired on the investment, i is the sinking fund rate and R_r is the total charge for both interest on book value and for depreciation for the rth year.

If $(1+i)^{r-1}$ be replaced by $(1+is_{\overline{r-1}|i})$ in (3), the expression for R_r may be written in terms of the R of (1) as

$$R_r = \left(i'F + \frac{W}{s_{\overline{n}|i}}\right) + (i - i')s_{\overline{r-1}|i} = R + (i - i')s_{\overline{r-1}|i}.$$

In this form it is evident that unless $i=i'$ the property owner will not be earning the desired rate i' on the investment as stipulated. Furthermore the rate actually earned will be greater than or less than i' according as $i>i'$ or $i<i'$, respectively. It is also clear that when $i'=i$, $R_r=R$.

3. An extension of the basic equation (1). Up to this point in the consideration of (1) it has been assumed that the payment period and interest conversion periods are all equal. Seldom in actual practice will such ideal conditions be encountered, consequently several examples of some complexity will now be

* J. H. Moore, Handbook of Financial Mathematics, New York, 1945, (181), p. 1041.

given to illustrate the versatility of (1) when interpreted logically.

Multiplying (1) by $s_{\overline{n}|i}$ and rearranging terms gives

$$F + Fi's_{\overline{n}|i} = S + Rs_{\overline{n}|i},$$

which simply states the equivalence between

(4) [*the total investment+the accumulated value of the desired income*] and [*the terminal value S+the accumulated value of the net income received*];

both accumulated values are obtained at the available reinvestment rate.

Two examples will now be given illustrating the versatility of (4).

Example 1. How much should be paid for the use of a certain property for 25 years which will yield no return for a period of 5 years, \$10,000 per year for 5 years commencing at the end of six years, followed by \$15,000 per year for the succeeding 10 years and \$25,000 for each of the last 5 years, if the investor will be required to make an expenditure of \$10,000 at the end of 25 years to restore the property to its original condition and it is desired that a rate $i^{(2)}=.06$ be earned on the investment? Assume that all income can be reinvested at $i^{(4)}=.03$.

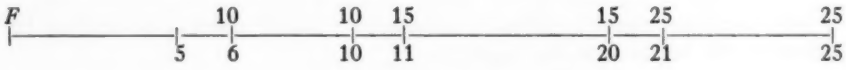

FIG. 1*

F+the accumulated value of the desired income is

$$F + (.03)Fs^{-1}_{\overline{2}|.0075}s_{\overline{100}|.0075}.$$

The accumulated value of the net income may be written

$$\$5{,}000\{2s^{-1}_{\overline{4}|.0075}s_{\overline{80}|.0075} + s^{-1}_{\overline{4}|.0075}s_{\overline{60}|.0075} + 2s^{-1}_{\overline{4}|.0075}s_{\overline{20}|.0075} - 2\}.$$

Equating and solving for F gives

$$F = \frac{\$5{,}000s^{-1}_{\overline{4}|.0075}\{2s_{\overline{80}|.0075} + s_{\overline{60}|.0075} + 2s_{\overline{20}|.0075}\} - \$10{,}000}{1 + (.03)s^{-1}_{\overline{2}|.0075}s_{\overline{100}|.0075}}$$

or $F=\$126{,}000$, to the nearest \$1,000.

Example 2. A man who pays \$100,000 for a piece of property which he keeps for 5 years and at the end of two years invests an additional \$50,000 desires to know what average interest rate he has earned per year on his investment if he sells it for \$20,000 at the end of 5 years and during the period has received a monthly income of \$3,000 per month for the first year, no income for the second

* The figures above the line are in thousands of dollars; those below the line, in years.

year and \$5,000 per month for the last three years, if it is assumed that money can be reinvested at $i^{(2)} = .025$ the first two years and at $i^{(2)} = .03$ during the last three years.

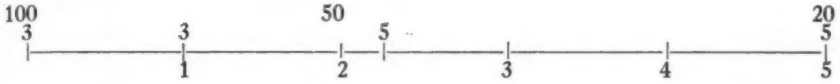

FIG. 2

$F+$the accumulated value of the desired income is

$$\$150{,}000 + \left\{ i'\$100{,}000\, s^{-1}_{\overline{2}|.0125} s_{\overline{4}|.0125} (1.015)^6 + i'\$150{,}000\, s^{-1}_{\overline{2}|.015} s_{\overline{6}|.015} \right\}.$$

The accumulated value of the net income at the end of 5 years may be represented as

$$\$1{,}000\left\{ 20 + 18 s^{(6)}_{\overline{1}|.0125} s_{\overline{2}|.0125} (1.0125)^2 (1.015)^6 + 30 s^{(6)}_{\overline{1}|.015} s_{\overline{6}|.015} \right\}.$$

Equating and solving for i' gives

$$i' = \frac{\$1{,}000\left\{ 18\, s^{(6)}_{\overline{1}|.0125} s_{\overline{2}|.0125} (1.0125)^2 (1.015)^6 + 30\, s^{(6)}_{\overline{1}|.015} s_{\overline{6}|.015} - 130 \right\}}{\$50{,}000\left\{ 2 s^{-1}_{\overline{2}|.0125} s_{\overline{4}|.0125} (1.015)^6 + 3 s^{-1}_{\overline{2}|.015} s_{\overline{6}|.015} \right\}}$$

or $i' = 14.4\%$, to the nearest tenth percent.

It would be pointless to try to give general formulas to fit all the situations which can arise; however, the two examples given serve to illustrate two types of problems which can be solved by (4) and will suggest to ingenious teachers a limitless number of variations.

MATHEMATICAL EDUCATION NOTES

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PROGRAM OF VISITING LECTURERS OF THE M.A.A.

ROTHWELL STEPHENS, Knox College

The Visiting Lectureship Program of the Mathematical Association of America has now completed its fifth year under support of the National Science Foundation. During the five years it has provided lecturers to 231 different

institutions, involving a total of 798 days of visitation. The increasing size of the program is shown by the number of days of lecturing each year and number of institutions visited.

	Number Days of Visits	*Number of Institutions*	*Average Length of Visit*
1954–55	169	66	2.6
1955–56	104	57	1.8
1956–57	169	81	2.0
1957–58	122	66	1.8
1958–59	234	136	1.7

The lecturers have been:

1954–55: Professor R. H. Bing, University of Wisconsin
Professor W. L. Duren, Tulane University (now at the University of Virginia)
Professor Tomlinson Fort, University of South Carolina
Professor George Polya, Stanford University
Professor D. V. Widder, Harvard University

1955–56: Professor D. H. Lehmer, University of California, Berkeley
Professor George Polya, Stanford University
Professor G. Baley Price, University of Kansas

1956–57: Professor Edwin Hewitt, University of Washington
Professor Kurt Mahler, University of Manchester
Professor A. W. Tucker, Princeton University
Professor J. L. Walsh, Harvard University

1957–58: Professor H. S. Coxeter, University of Toronto
Professor Paul Halmos, University of Chicago
Professor John Kemeny, Dartmouth College
Professor Walter Prenowitz, Brooklyn College
Professor R. D. Schafer, University of Connecticut

1958–59: Professor Tom M. Apostol, California Institute of Technology
Dr. Robert E. Gaskell, Boeing Airplane Company
Professor R. E. Johnson, Smith College
Professor John L. Kelley, University of California, Berkeley
Professor Marguerite Lehr, Bryn Mawr College
Professor Lynn H. Loomis, Harvard University
Professor S. S. Wilks, Princeton University
Dr. Stanislaw Ulam, Los Alamos Scientific Laboratory

1959–60: Professor David Blackwell, University of California, Berkeley
Professor J. Sutherland Frame, Michigan State University
Professor Magnus Hestenes, University of California, Los Angeles
Professor S. A. Jennings, University of British Columbia
Professor Tibor Rado, Ohio State University
Professor Ernst Snapper, Indiana University

Mathematicians, the M.A.A., and the Committee are greatly indebted to these men for the time, energy, and talent which they have given to the program. It is their contribution which has made the program outstandingly successful, and they have our sincere appreciation.

In spite of the belief of the Committee on Visiting Lecturers that visits of three or more days would be optimum and in spite of the increase in lecturing time available, the growth of the demand for lecturers has continued to lower the average number of days spent at each institution, the average of 1.7 for 1958–59 being the minimum. The time requests of many institutions were reduced, and a good number of requests were not granted. A reasonable estimate of the total audience reached in 1958–59 is 25,000, counting duplications.

The program for 1959–60 was oversubscribed. Among the applications there were a large number from schools which had never before participated in the program. This means that further increases in the size of the project will occur. To cope with this increase the Committee has been enlarged, the current membership being: Burton W. Jones, University of Colorado; Paul B. Johnson, U.C.L.A.; R. C. Fisher, Ohio State University; R. E. Gaskell, Oregon State College; R. E. Johnson, Smith College; Robert Rosenbaum, Wesleyan University; C. L. Seebeck, Jr., University of Alabama; and, as chairman, Rothwell Stephens, Knox College.

In order to accommodate more institutions and to establish closer relations between the mathematics departments in the universities and the neighboring colleges, a Visiting Mathematicians Program was inaugurated this year. Interested universities have nominated members of their staffs to serve as Visiting Mathematicians to nearby colleges. The Visiting Mathematician functions in much the same way as the Visiting Lecturer.

The Program of Visiting Lecturers of the Mathematical Association was the first lecturing program undertaken on a nationwide basis in the sciences, and it would not have been possible without the generous support of the National Science Foundation. With the support of the Foundation, the Program is assured through 1960–61. The enthusiastic reports received from institutions and from lecturers indicate that it is a program which should be continued on a permanent basis.

A SUMMER INSTITUTE ON THE TEACHING OF ARITHMETIC FOR ELEMENTARY SCHOOL PERSONNEL

JOSEPH N. PAYNE, University of Michigan

Under the sponsorship of the National Science Foundation, the University of Michigan held a four-week institute, July 20 to August 15, 1959, on the teaching of arithmetic for personnel of the elementary school. Of the fifty-two participants in the institute, 20 were supervisors or consultants, 15 were principals or superintendents with supervisory functions, 13 were elementary school teachers, 2 were junior high school teachers, and 2 taught arithmetic method courses in college.

Purpose of the institute. An understanding of and appreciation for arithmetic is a first step in achieving mathematical and scientific literacy needed by

citizens of the modern world of today. That teachers may help children to learn arithmetic with meaning and understanding, teachers must: (1) know the fundamental mathematical ideas; (2) know the principles of psychology and their implications for teaching arithmetic; and (3) be able to translate these ideas into effective teaching in the classroom. The Institute was designed to improve instruction in arithmetic by helping teachers and supervisory staff improve their competency in the three areas listed above.

The program. The program designed to help achieve the above-named objectives consisted of the following:

1. A mathematics course in the foundations of arithmetic: This course presented the logical structure of the number system and the basic mathematical concepts of arithmetic. Special attention was given to the historical background of the development of these ideas.

2. An education course in the psychology and teaching of arithmetic: This course presented the basic concepts of psychology and human growth and development as they relate to the development of mathematical concepts by children. Guest lecturers and demonstration classes helped in bridging the gap between theory and practice.

3. A daily period for supervised reading, discussion, and writing: Each participant prepared some written materials—curriculum outlines, sample teaching units, suggestions for teaching aids and materials, or plans for in-service education of teachers. These were duplicated and given to each participant. A special library of texts, supplementary books, curriculum guides and teaching materials was available for use by participants.

4. Lectures and demonstration classes: Special lectures dealt with recent experimentation, revision of content, current trends, and curriculum organization in arithmetic education as well as psychology and human growth and development. The demonstration classes included one closed-circuit TV demonstration.

The staff. The staff members of the Institute consisted of: Professor E. Glenadine Gibb (Associate Director), Iowa State Teachers College; Professor Phillip S. Jones, The University of Michigan; Dr. Edwina Deans, Elementary Supervisor, Arlington County, Virginia, Public Schools; Miss Alice Hach, Mathematics and Science Consultant, Racine, Wisconsin, Public Schools; Professor Joseph N. Payne (Director), The University of Michigan.

THE CASE FOR BASIC RESEARCH ON THEORY OF INSTRUCTION

GERTRUDE HENDRIX, University of Illinois

Theory of instruction research now under way seeks an underlying explanation of the teaching approach most successful with courses developed by the UICSM Mathematics Project. The University of Illinois Nonverbal Awareness

Project, supported initially by small private grants and directed by the Teacher Coordinator of the UICSM Project, is now operating at a hypothesis-generating level only. Facilities have been relinquished and equipment stored for a two-year period in order to release research personnel for intensive work on a film project, the National Educational Television and Radio Center Mathematics Study.

The primary aim of the film project is to produce a series of films to replace the demonstration classes which are now an essential accompaniment to UICSM teacher-training programs. Incidentally, however, the films also are providing a rich store of recorded events from which data for the underlying theoretical study can proceed two years hence. Footage already obtained contains many vivid illustrations of the kind of learning the new courses seek to promote.

One phase of teaching looming large in things revealed to date is the enormous role played by nonverbal communication between teacher and students. Current research in paralinguistics—especially that involving kinesics—is revealing ways to identify and classify the nonverbal behavior which human beings learn to interpret in each other; it is thus that we produce the complicated stream of communication sometimes accompanying, sometimes independent of, words. Such an analysis might enable a teacher to cultivate desirable paralinguistic effects, and to avoid those which are destructive to his work.

Effective teacher-training films must help teachers learn to *observe* students. Sympathetic observation is prerequisite for clues as to what to say or do next in order to make learning "come clear." Since it was a recent re-interpretation of animal learning which led to identification of the nonverbal stage in creative abstract human learning, some have suggested that observations of animals in training might speed up acquiring ability to tell when someone has "caught on" to an idea. Two of the twenty films in the NETRC Math Study series will contain shots for testing this kind of observation power.

Those involved in the nonverbal awareness research at the University of Illinois feel that a re-examination of the entire learning process is in order. Even the simplest so-called conditioned reflex can be interpreted as evidence that the organism possesses a generalization of a type the logicians call 'universal conditional'.

Phenomena which indicate the advent of awareness are of basic importance in teaching for discovery, and all such phenomena are alike in that they consist of behavior that would have been impossible without awareness of the concept or generalization being taught. Usually these phenomena do *not* include a verbal formulation of the thing just discovered. In fact, in a creative sequence it is awareness of a new concept which impels one to seek or to invent a name for it.

Electroencephalographs of animals acquiring conditioned reflexes reveal decided changes in neutral activity. We wish that we could see brain-wave records of the seventh-graders who were becoming aware of the distributive principle for multiplication over addition in NETRC Math Study Film Number IX.

A REPORT ON THE BATON ROUGE CENTER FOR GRADES SEVEN AND EIGHT SCHOOL MATHEMATICS STUDY GROUP, 1958–59

HOUSTON T. KARNES, Louisiana State University

The Baton Rouge Center was one of twelve areas invited to engage in the 1958–59 experimental program for grades seven and eight. The other eleven centers were: University of Arizona; University of Chicago; University of Colorado; University of Delaware; University of Michigan; University of Minnesota; Brookline, Massachusetts; Pasadena, California; Princeton, New Jersey; Seattle, Washington; Westport, Connecticut.

Center reports indicate that the experience in Baton Rouge was typical of those in the other centers. A center was composed of seventh and eighth grade teachers, a chairman and a consultant. The chairman could be a person in the field of teacher training, a supervisor in mathematics, one from the field of mathematics education, or a college or university teacher of mathematics well versed in secondary school work. The consultant was a mathematician from the staff of a college or university. The Baton Rouge Center was composed of four seventh grade teachers, and two eighth grade teachers. The author doubled as the chairman and the consultant. The teachers were all duly certified to teach junior high school mathematics, which required a mininum of eighteen semester hours in mathematics. Only two had more than the minimum and none of the teachers had had recent courses in college mathematics. The college work of these teachers, in mathematics, was of the traditional type, and therefore did not include certain vocabulary, ideas and concepts these teachers would be using in the experimental material. Each teacher taught three classes of the SMSG seventh grade text and two or three classes using a traditional text. This gave a total of eighteen experimental classes involving some 640 pupils. The teachers were drawn from schools of various sections of the city so that there was a good socio-economic class distribution among the pupils. The pupils were not a select group; the I.Q. range was from well below average to superior. In only one school was there any attempt at homogeneous grouping.

Program of the center. Once per week, throughout the year, a two to three hour seminar was held. On several occasions it was found desirable to meet twice per week. Most of the time of these meetings was devoted to studying the text, introducing related mathematical ideas which would help the teachers, and in discussing the presentation of the text material. Time was allotted for the raising and answering of questions and also in discussing classroom experiences of the teachers. In addition, part of each seminar was devoted to the testing program.

At the conclusion of each unit of the text all teachers submitted unit reports. These reports consisted of a questionnaire, a page or two of additional comments by the teacher, and a copy of the unit test with the test results. These reports were of distinct aid to the writing team during the summer of 1959. The last part of the program dealt with the preparation of new materials and editing the

current materials. Each center was encouraged to either write new units or rewrite one of the existing units. In addition, each center was asked to edit one unit of the text as preparations were being made for a new printing.

Reaction to the program.

1. *The teachers.* The teachers were highly enthusiastic with respect to the materials used and to student reaction. In general the teachers found most all of the material used to be teachable. They found this approach to be a good motivating vehicle. Several teachers made comments to the effect that some below-average students who were taking no interest in the traditional material became alive and really took an interest in the activities of the class. This was even more noticeable among the above-average students.

Although no actual study was made, from time to time the teachers gave the experimental classes the same tests that were given to the traditional classes. In general it was found that the experimental classes did better on these tests than did the pupils of the traditional group. One teacher gave standardized achievement tests to all of his pupils in the spring of 1959. The result showed that the experimental classes were accelerated from one-half year to one and one-half years beyond the traditional classes.

2. *The pupils.* The new approach of the SMSG text was well received by the pupils. This seemed to be due to the following reasons:

1) Material which was new and different from the uninteresting traditional arithmetic of the first six years—arithmetic which has been a drudgery to many pupils and has caused many good pupils to be lost to the field of mathematics.

2) In the SMSG text the pupils found answers for previous perplexing questions and reasons for technique which were being developed.

3) The enthusiasm of the teachers.

4) The interest on the part of their parents.

Space does not permit quotations from the pupils; however, many made statements of approval with regard to the new material. Only seven pupils in the experimental classes requested to be changed to a traditional class and three of these were due to parental influence, probably because those parents found it difficult to help their children in this new approach.

3. *The parents.* In some sections of the city there were no parental comments at all. This, however, was to be expected. In those areas where professionally trained parents and parents who were business leaders lived the interest ran high. Many of these parents learned right along with their children. Some even requested permission to visit the classes. On parent visitation nights the teachers in the program would be swamped with visitors. A large number of parents, having heard of the program, made requests that their children be transferred to the experimental classes. These requests became so large and demanding last fall (1959), that it became necessary to add five new classes in one of our schools.

REACTION TO THE KANSAS TEPS CONFERENCE

W. V. PARKER, Auburn University

The Kansas Conference in June 1959 was organized in very much the same way as the 1958 Bowling Green Conference. (This MONTHLY, vol. 65, pp. 698–700 and vol. 66, pp. 59–60.) The approximately 1200 participants were divided into 34 groups designated as Set I and then were regrouped into 30 groups designated as Set II. The groups of Set I met on Wednesday to discuss the general problems of teacher training. In each group a report was given on the teacher training program of a selected institution.

The groups of Set II met on Thursday to discuss special training for the various areas. The ones relating to the training of elementary teachers were divided into several levels and those concerned with secondary teaching were divided according to subject matter areas. In some cases there were several groups concerned with the same area. There were two groups on mathematics. These groups were concerned with the training needed for their special areas.

The groups of Set I reassembled on Friday to attempt to bring together the ideas of the various groups of Set II and to formulate a pattern for teacher training at the various levels.

One was impressed by the wide-spread academic interest of the participants. Apparently there was a much larger representation from the nonprofessional areas than has been the case at previous meetings.

The need for strong subject matter training for teachers at all levels was emphasized in the discussions generally. Much of the controversy of academic versus professional work seemed to be giving way. There seemed to be general agreement that the training of teachers must be a college-wide problem and not just the problem of the schools of education.

The willingness of both sides to make concessions is encouraging. Other meetings of this kind on both regional and national levels should prove to be quite worthwhile.

MATHEMATICS COURSES FOR ELEMENTARY TEACHERS

One of the most important and difficult problems in mathematics teacher education is at the elementary school level. While there is agreement that mathematics should be required of all elementary teachers, there is no general agreement on what the content and nature of the course or courses should be. In the April issue Lenore John reported on the geometry content needed by the elementary teacher. The editors of this department have invited a number of departments of mathematics to contribute descriptions of courses in mathematics which they offer for elementary teachers. Below are reports received in response to this invitation from a liberal arts college, a teachers college, a privately endowed university, and four state universities.

Emory University. Emory currently has two mathematics courses for elementary school teachers. Both deal with the mathematical foundations of arithmetic and its relation to elementary school instruction; both have two instructors, one from Mathematics and one from Education. One course, sponsored by the National Science Foundation,

is for in-service teachers and is meeting once a week for 30 two-hour sessions. The second course, supported by the American Association for the Advancement of Science, was offered in the winter quarter to undergraduates preparing for teaching.

In both courses the mathematics instructor is responsible for the organization and presentation of the mathematical content and the education instructor is responsible for problems of application and interpretation in elementary school teaching and for relevant material on the psychology of learning and teaching. Actually it has turned out that both instructors have shared extensively in all tasks. It should be noted that a careful delineation for responsibility in such a "team-teaching" situation is nevertheless probably a sound precaution.

For the in-service course a graduate assistant, who has had school teaching experience and graduate courses in both Mathematics and Education, has been devoting 20 hours a week to visiting the teachers in their classrooms. This assistant has made a valuable contribution to the individual instruction and guidance for the teacher and has performed the important function of giving the teachers direct assistance in applying course material in their teaching.

Bevan K. Youse

Illinois State Normal University. The Mathematics Department at Illinois State Normal University offers a basic program in mathematics education for teachers of the elementary grades in two courses. The first course, usually taken in the freshman year, is primarily a content course, Basic Concepts of Arithmetic, which emphasizes those concepts growing out of counting and numbers as well as those growing out of measuring. This course gives credit of three semester hours. The second course, Arithmetic for the Elementary Grades, for two semester hours emphasizes the problems of learning and teaching elementary mathematics. This work is usually taken by juniors before they do their student teaching. Most of the students are doing junior participation in the elementary classrooms at this time.

Students preparing to teach at the junior high level are enrolled in a course called Mathematics for the Junior High School Grades. About half of these students are mathematics majors or minors and the remainder are elementary majors who have strong mathematics background from high school and have completed the five semester hours program for the elementary curriculum. Advanced work is offered in three graduate courses—Teaching of Arithmetic, Teaching of Junior High School Mathematics, and Teaching and Supervision of Mathematics. Throughout the program, mathematical content and significant historical developments are stressed.

Francis R. Brown

Knox College. Students majoring in elementary education are required to take six semester hours in the Mathematics Department and three semester hours in the teaching of arithmetic. It is usually recommended that students take a year course, Basic Mathematics, to meet the mathematics requirement. This course is designed for students who have had only two years of college preparatory mathematics. The course includes an axiomatic approach to the natural numbers, the integers, the rational numbers, the real numbers, and the complex numbers. Also discussed are prime numbers, induction, the decimal system, number scales, arithmetic and geometric progressions, linear equations, systems of linear equations, square roots, quadratic equations, mathematical systems, groups, isomorphism, miniature geometries, graphs and functions. The course concludes with brief introductions to trigonometry, analytic geometry, and calculus. The Teaching of Arithmetic is taught in the Department of Education, currently by the Director of Elementary Education in the local public school system.

Rothwell Stephens

Montana State College. At Montana State College two courses in mathematics are required of students majoring in Elementary Education. The two courses are: Intermediate Mathematics, 4 credits, and Teaching of Arithmetic, 3 credits.

In the first part of the course in Intermediate Mathematics, a review of arithmetic operations with ordinary and decimal fractions is given. Stress is placed on our number system, as well as numbers in some scale other than ten. Work is included on denominate numbers and the metric system. The second part of the course includes a review of elementary algebra. Emphasis is placed on exponents, evaluation of formulas, the fundamental operations with algebraic fractions, and the solution of linear equations.

In the Teaching of Arithmetic, which is a junior level course, emphasis is put on the understanding and mastery of the usual operations in arithmetic. Some attention is given to the methods of teaching the usual operations. The need for reading current periodicals on the teaching of arithmetic and the desirability of a mathematics library at the elementary level is pointed out.

Adrien L. Hess

University of California, Los Angeles. Fundamentals of Arithmetic. (3) I, II. Prerequisite: sophomore standing. Designed primarily for prospective teachers of arithmetic. The study of the fundamental operations on integers and fractions is stressed, together with suitable visual aids. Although efficiency in arithmetical skills is required, the emphasis is on the understanding of arithmetical procedures.

This course is required for all elementary credential candidates and furthermore, is highly recommended for general secondary candidates.

Clifford Bell

University of Maryland. In considering material for a mathematics course suited to the needs of students preparing to teach in the elementary schools, primary attention was given to those topics which would provide deeper understanding of the foundations of the arithmetic taught in the elementary grades. Accordingly, topics in the algebra of integers and of rational numbers, and in the elementary theory of numbers were selected. Included in the course are the following:

1. Inductive proof
2. The system of natural numbers, based on the Peano Axioms
3. Mathematical systems: Elements, operations, properties
4. The algebraic structures, group and field, modular arithmetic
5. The system of integers: Ordered pairs of natural numbers
6. The system of rational numbers: Ordered pairs of integers
7. Congruences and rules for divisibility
8. Euclidean algorithm for the greatest common factor
9. Number system on bases other than ten.

This is a four semester-hour course, the students meeting for one-hour sessions, four days a week.

Clearly, many of these topics receive a sophisticated treatment in advanced courses. At this level, while the rigor of the proofs is maintained, the presentation of the material is made from a point of view which requires only the skills acquired in a first course in high school algebra. It has been our experience that the students are sufficiently mature to grasp the concepts of the topics outlined above, if properly presented. Carefully planned developmental exercises help surmount the difficulty of limited manipulative algebraic skill. Upon completion of the course, the student is able to read with understanding, current literature in "new" mathematics for the pre-high school student.

Helen L. Garstens

The University of Oklahoma. The University of Oklahoma is offering a course in the fundamentals of arithmetic which is designed for the pre-service preparation of the elementary teacher. The primary aim of this course is to emphasize the structure of our numbers, beginning with the natural numbers and terminating with the real numbers as a field. The historical development of these numbers and the numerals used to denote them is studied with special consideration given to their contribution to our culture.

Elementary set theory is used to give a deeper understanding of the operations and the laws governing these operations. It is used as a unifying concept in the teaching of arithmetic at the elementary level. Accurate but beginning interpretations are given to many concepts which appear in more advanced courses, for example, the concept of variable. This is introduced through the study of various types of sentences. Some time is also given to the study of numeration in order that a better understanding of computational skills may be acquired.

Eunice Lewis

California Conference on Secondary School Mathematics

One of the most encouraging signs of the great progress which is being made in the improvement of mathematics instruction in the secondary schools and the important relationships of mathematicians with school administrators and educationists is given by the Conference on Secondary-School Mathematics recently held at the University of California, Santa Barbara, under the sponsorship of the California State Department of Education. When a state department of education arranges a conference to consider secondary-school mathematics curriculum and is able to bring together such outstanding mathematicians as E. G. Begle, Karl Menger, and R. L. Wilder, in addition to quite a number of leading mathematicians from California including John L. Kelley and Harold M. Bacon, then we in mathematics have cause for real optimism about present improvement of secondary instruction. Such a conference as this should remove, on the part of many, the fear that is sometimes heard that the mathematicians have not sufficiently sought the cooperation and assistance of secondary-school administrators and teachers, and hence that the product of the various mathematics curriculum studies, particularly the School Mathematics Study Group, will suffer therefrom in acceptance. The large numbers of people representing various interests that are involved in the School Mathematics Study Group in the try-out of materials in more than 100 schools throughout the country is largely responsible for this happy turn of events.

J. R. Mayor

Science Education News

The Conference Board of the Mathematical Sciences has named Dr. Leon W. Cohen, head of the department of mathematics of the University of Maryland, as Contributing Editor for *Science Education News.* This is a quarterly publication issued by the Science Teaching Improvement Program of the American Association for the Advancement of Science.

Science Education News reports science education activities of interest to scientists as members of scientific societies and in industrial and government positions. Interested persons may have their names added to the mailing list by writing to Mrs. Charlotte W. Colton, Editor, Science Education News, 1515 Massachusetts Avenue, N.W., Washington 5, D. C.

In the December 1959 issue of *Science Education News,* the editorial is written by Dr. Cohen.

ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY HOWARD EVES, University of Maine

Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, University of Maine, Orono, Maine. This department welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 1416. *Proposed by Leonard Cohen, City College of New York*

Prove that $x^2+y^2=z^n$, where n is a positive integer, always has nonzero integral solutions.

E 1417. *Proposed by Robert Hartop, Los Angeles, California*

Given a unit circle with point P on its circumference and a distance d_1, $0<d_1<2$. With center at P, construct the circle of radius d_1 to cut the given circle in two points. If d_2 is the distance between these points, construct a new circle of radius d_2 with center at P and so obtain $d_3, d_4, \cdots, d_n, \cdots$. Find $\lim_{n\to\infty} d_n$.

E 1418. *Proposed by R. C. Buck, Institute for Defense Analyses, Princeton, New Jersey*

In *Comptes Rend.* (1945) vol. 62, pp. 95–97, there appears the theorem: If $f(x, y)$ has partial derivatives of first and second orders, and if $f(a, b)+f(b, c)=f(a, c)$ for all a, b, c, then there is a function ϕ such that $f(x, y)=\phi(x)-\phi(y)$. Can these hypotheses be weakened?

E 1419. *Proposed by C. C. Yalavigi, Government College, Mercara, India*

In the *Collected Papers of Srinivasa Ramanujan*, edited by G. H. Hardy, Seshu Aiyar, and B. M. Wilson (1928), appears (p. 334, Question 1076 (XI, 199)) the following:

Show that

(i) $$[7(20)^{1/3}-19]^{1/8}=(5/3)^{1/3}-(2/3)^{1/3},$$

(ii) $$[4(2/3)^{1/3}-5(1/3)^{1/3}]^{1/6}=(4/9)^{1/3}-(2/9)^{1/3}+(1/9)^{1/3}.$$

Show that these relations are incorrect, and that to correct them we must interchange the exponents 1/8 and 1/6 appearing in the left-hand sides.

E 1420. *Proposed by Victor Thébault, Tennie, Sarthe, France*

Let A', B', C' (A'', B'', C'') be the centers of squares described exteriorly (interiorly) on the sides BC, CA, AB of a triangle ABC. Show that the radical center of the circles $A(A')$, $B(B')$, $C(C')$, ($A(A'')$, $B(B'')$, $C(C'')$) coincides with the nine-point center of triangle ABC.

SOLUTIONS

Two Trains

E 1386 [1959, 812]. *Proposed by Monte Dernham, San Francisco, California*

In overtaking a freight, a passenger train which is x times as fast takes x times as long to pass it as it takes the two trains to pass when going in opposite directions. Find x.

Solution by Underwood Dudley, University of Michigan. If the combined length of the trains is d and the speed of the freight is v, then the conditions give $d/v(x-1)=xd/v(x+1)$. Keeping the positive root we find $x=1+\sqrt{2}$.

Also solved by A. N. Aheart, R. G. Albert, M. H. Alief, J. W. Baldwin, Leon Bankoff, Edward Barbeau, Robert Bart, J. F. Barth, Alan Beal, W. R. Becker, R. E. Blewster, Jr., Karl Bochert and Laurie Dodge (jointly), D. A. Breault, E. W. Brown, Charles Bush, Marcus Charles, I. Chorneyko and W. Zayachkowski (jointly), A. G. Clark, N. A. Court, F. T. Driscoll, J. M. Elkin, G. W. Erwin, Jr., Irma Esrig, Arthur Evans, Jr. and H. R. van Zoeren (jointly), W. D. Evans, Walter Feibes, Geoffrey Fox, Stuart Friedman, Susan L. Friedman, R. E. Glenn, Michael Goldberg, L. D. Goldstone, C. L. Goode, A. G. Grace, Jr., S. H. Greene, Newcomb Greenleaf, Spencer Hamilton, Ned Harrell, Dom Dunstan Hayden, E. E. Heimann, J. H. Hodges and D. B. Larson (jointly), Richard Holt, E. L. Hubbard, W. O. Hulser, A. R. Hyde, J. N. Issos, Gerald Janusz, Kim Jones, J. P. Jordan, A. F. Kaupe, Jr., R. W. Kilmoyer, J. M. Kingston, P. G. Kinney, Jr., Leonard Klafter, Donald Knuth, Frank Kocher, Sidney Kravitz, M. E. Lakser, W. E. Lawrence, Ruth Lemon, Joel Levy and H. J. Lieberman (jointly), Norton Levy, R. J. Lewyckyj, L. I. Lowell, J. R. Lux, Joseph McMeudie, Barry MacKichan, Wallace Manheimer, D. C. B. Marsh, Vincent Mead, J. W. Mettler, N. H. Mines, Morris Morduchow, C. S. Ogilvy, Bart Park, P. M. Pepper, J. L. Pietenpol, C. F. Pinzka, G. S. Plews, Thomas Porsching, D. K. Richards, L. A. Ringenberg, Mary J. Robertson, D. R. Rogers, M. T. Salhab, C. M. Sandwick, Sr., E. M. Scheuer, L. J. Schneider, R. T. Shannon, L. C. Shepley, Harold Shulman, J. K. Siberz, S.J., C. M. Sidlo, D. L. Silverman, Arnold Singer, Vencil Skarda, E. L. Spitznagel, Jr., C. L. Sterling, D. C. Stevens, P. C. Stewart, W. B. Stovall, Jr., Melvin Tainiter, Hoyt Taylor, C. W. Trigg, Chung-lie Wang, Larry Webster, W. E. Wiebenson, Ralph Wiggins, Donald Williams, C. Winston, Tim Winterer, G. L. Woodruff, C. C. Yalavigi, A. W. Yonda, I. Zucker, and the proposer. Late solutions by John Burslem, George Glauberman, J. B. Muskat, and D. J. Peterson.

A Line Dividing the Area Under a Sine Arch

E 1387 [1959, 812]. *Proposed by L. T. Van Tassel, San Diego, California*

Find p such that the line $y=p$ divides the area of an arch of the sine curve $y=\sin x$ into parts in a given ratio to one another.

Solution by A. R. Hyde, Bronxville, N. Y. The same ratio r may be applied to the half-arch of $y=\cos x$. Then

$$\int_0^p \arccos y\,dy = r/(r+1),$$

giving the equation

$$(1-p^2)^{1/2} - p \arccos p = 1/(r+1).$$

This may also be written $\sin t - t\cos t = 1/(r+1)$, where $\cos t = p$.

Also solved by A. N. Aheart, Robert Bart, D. A. Breault, E. W. Brown, Louise Grinstein,

E. L. Hubbard, Donald Knuth, Wallace Manheimer, D. C. B. Marsh, Vincent Mead, E. M. Scheuer, Arnold Singer, Larry Webster, and C. C. Yalavigi. Late solution by Kenneth Stolarsky.

Yalavigi showed that if we set $k=1/(r+1)$, then we have, approximately,

$$p^3 = [(1+k^5)^3 - 4k^3]/2(3+k^5).$$

A System of Inequalities

E 1388 [1959, 813]. *Proposed by H. W. Gould, West Virginia University*

Determine the largest integral value of n such that there exists a solution to the system of inequalities

$$k < x^k < k+1, \qquad k = 1, \cdots, n.$$

Would it make any difference if $<$ were replaced by $\leqq$?

Solution by N. J. Fine, Institute for Advanced Study. The largest possible n is 4. If $n \geqq 5$, then the inequalities $3 \leqq x^3$ and $x^5 \leqq 6$ would imply $3^5 \leqq 6^3$, which if false. For $n=4$, any number x satisfying $3^{1/3} < x < 5^{1/4}$ will do. Changing $<$ to $\leqq$ makes no difference.

Also solved by R. G. Albert, J. W. Baldwin, Edward Barbeau, Robert Bart, D. A. Breault, E. W. Brown, Underwood Dudley, J. W. Ellis, Arthur Evans, Jr. and H. R. van Zoeren (jointly), Geoffrey Fox, S. P. Franklin and G. A. Hutchison (jointly), Leo Gafney, S.J., Michael Goldberg, S. H. Greene, J. H. Hodges, J. Hooley, A. R. Hyde, Gerald Janusz, J. P. Jordan, Donald Knuth, Sidney Kravitz, Joel Levy and H. J. Lieberman (jointly), R. J. Lewyckyj, Wallace Manheimer, D. C. B. Marsh, C. S. Ogilvy, Walter Penney, P. M. Pepper, J. L. Pietenpol, C. F. Pinzka, Thomas Porsching and R. H. Shaw (jointly), L. A. Ringenberg, L. C. Shepley, J. K. Siberz, S.J., D. L. Silverman, Vencil Skarda, C. D. Stevens, J. E. Vinson, W. E. Wiebenson, Donald Williams, and the proposer. Late solutions by George Glauberman and J. B. Muskat.

Franklin and Hutchison generalized by showing that if $\phi(m)$ is the maximum number of solvable inequalities

$$k < x^k < k+1, \qquad k = m, m+1, \cdots, n,$$

then for $m=1, 2, \cdots, 6$, $\phi(m)$ takes on the values 4, 3, 2, 2, 2, 1, and for $m \geqq 7$, $\phi(m)=1$.

Evaluation of a Determinant

E 1389 [1959, 813]. *Proposed by E. M. Scheuer, Space Technology Laboratories, Los Angeles, California*

Evaluate the $n \times n$ determinant A_n whose (i, j)th entry is $a^{|i-j|}$.

I. *Solution by E. E. Strock, Prudential Insurance Company, Newark, N. J.* Expand the determinant A_n by the elements of the first row. The minor of the first element is A_{n-1}; the minor of the second element is aA_{n-1}; the minors of the remaining elements of the first row are all zero. Hence $A_n = (1-a^2)A_{n-1}$. Since $A_1 = 1$, $A_n = (1-a^2)^{n-1}$.

II. *Solution by J. W. Ellis, Louisiana State University in New Orleans.* By subtracting a times the second row from the first row, we see that $A_n = (1-a^2)A_{n-1}$. Since $A_1 = 1$, it follows that $A_n = (1-a^2)^{n-1}$.

III. *Solution by T. A. Porsching, Fort Belvoir, Va.* By multiplying the

$(i-1)$st row by a, and subtracting it from the ith row, $i=n, n-1, \cdots, 2$, A_n is reduced to triangular form with $a_{11}=1$, $a_{ii}=1-a^2$ when $i\neq 1$. Therefore $A_n=(1-a^2)^{n-1}$.

Also solved by A. N. Aheart, R. G. Albert, Edward Barbeau, Robert Bart, Alan Beal, W. R. Becker, D. A. Breault, E. W. Brown, R. F. Brown, I. Chorneyko and W. Zayachkowski (jointly), Richard Cottle, Underwood Dudley, E. S. Eby, Arthur Evans, Jr., N. J. Fine, S. P. Franklin and G. A. Hutchison (jointly), Stuart Friedman, Michael Goldberg, Leonard Goldstone, H. W. Gould, S. H. Greene, Newcomb Greenleaf, J. H. Hodges, Vern Hoggatt, J. Hooley, A. R. Hyde, Gerald Janusz, J. P. Jordan, A. F. Kaupe, Jr., J. M. Kingston, Donald Knuth, Sidney Kravitz, W. E. Lawrence, Gerald Leibowitz, R. J. Lewyckyj, Wallace Manheimer, D. C. B. Marsh, P. T. Mielke, N. H. Mines, E. R. Mullins, Jr. and David Rosen (jointly), C. S. Ogilvy, F. D. Parker, J. L. Pietenpol, C. F. Pinzka, L. A. Ringenberg, R. T. Sandberg, C. M. Sandwick, Sr., L. J. Schneider, A. L. Schreiber, Donna Seaman, Harold Shulman, C. M. Sidlo, D. L. Silverman, Benjamin Sims, Arnold Singer, A. Sinkov, Vencil Skarda, J. R. Smart, E. L. Spitznagel, Jr., D. C. Stevens, J. R. Swenson, Chester Tsai, L. T. Van Tassel, Chung-lie Wang, W. E. Wiebenson, R. H. Wilson, Jr., Dale Woods, A. W. Yonda, S. M. Yusuf, David Zeitlin, the proposer, and an anonymous solver. Late solutions by J. W. Baldwin, H. F. Bechtell, C. H. Cunkle's *Modern Algebra* class, George Glauberman, Joseph Hammer, P. G. Kirmser, M. S. Klamkin, J. B. Muskat, D. J. Peterson, and B. E. Rhoades.

Hodges suggested, as an allied problem, the evaluation of the $n\times n$ determinant B_n whose (i, j)th entry is 0 for $i=j$ and $a^{|i-j|}$ for $i\neq j$. S. H. Eisman proposed the problem of finding the inverse of the $n\times n$ matrix whose (i, j)th entry is $a^{|i-j|}$ and where $|a|\neq 1$.

Primitive Decompositions

E 1390 [1959, 813]. *Proposed by J. Łoś and J. Mycielski, Wroclaw, Poland*

A *decomposition* of a rectangle (rectangular parallelepiped) R is a dissection of R into smaller rectangles (rectangular parallelepipeds) by lines (planes) parallel to the sides (faces) of R. If a decomposition of R is not a refinement of some other decomposition of R, then the decomposition is said to be *primitive*.

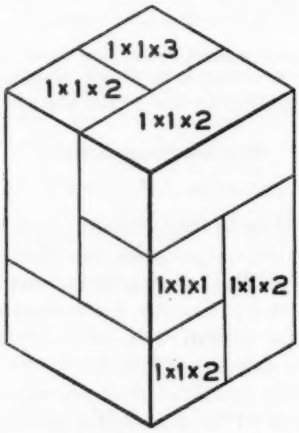
(1) Show that there exist primitive decompositions of a rectangle into any number $n>1$ of rectangles, except $n=3, 4, 6$.

(2) Show that there exist primitive decompositions of a rectangular parallelepiped into any number $n>1$ of rectangular parallelepipeds, except $n=3, 4$.

Composite solution by D. C. B. Marsh, Colorado School of Mines, and Michael Goldberg, Washington, D. C. Let us call any of the smaller rectangles into which the given rectangle is decomposed, a *part*; those parts with a vertex in common with the given rectangle will be called *corner parts*. Necessary conditions for a primitive decomposition with $n>2$ are (1) no two parts may have a common side, and (2) no corner part may have two vertices in common with the given rectangle. (The proof of (1) is that this common side, if removed, would result in a decomposition of $n-1$ parts, whence the original decomposition would be nonprimitive contrary to assumption; for (2), any decomposition with a corner part covering two vertices would be a refinement of a decomposition involving merely this part and the remaining portion of the given rectangle: i.e., a refinement of some $n=2$.) Thus we conclude there are at least four parts (each a distinct corner part) in any primitive decomposition having $n>2$. Thus $n\neq 3$.

For $n=4$, the interior vertices of the four corner parts would have to coincide, but such a decomposition obviously is not primitive. For $n>4$, the interior vertices of the four corner parts may be vertices of a rectangle, in which case we have a primitive decomposition with $n=5$. If the interior vertices of the four corner parts are not vertices of a rectangle, then the simplest remaining area is L-shaped and the least possible number of parts is 6. However, for the L-shaped configuration, the L may be broken into just two parts in only two ways, either of which results in a 6-part decomposition which is not primitive.

The following method may be used to construct primitive rectangular decompositions for $n=5, 7, 8, 9, \cdots$. Starting from a point (not a vertex) on one side of the given rectangle, draw an $(n-1)$-segmented rectilinear spiral with its segments parallel to sides of the rectangle and with the last "loop" closed; extend each segment (after the first) backward until it meets an orthogonal segment; there results a primitive decomposition with n parts.



For rectangular parallelepipeds, a strictly analogous argument involving the necessary condition that no corner part may have three (or more) vertices in common with the given parallelepiped and yield a primitive decomposition shows that $n=3, 4$ are not primitive. Each primitive decomposition for $n=5, 7, 8, 9, \cdots$ of a plane rectangle produces a primitive decomposition into the same number of parts of a rectangular parallelepiped by making the solid parts the same height in the third dimension. However, the primitive decomposition into 6 parts is also possible as is shown in the accompanying figure.

Also solved by Newcomb Greenleaf, Donald Knuth, and the proposers.

ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE, Rutgers, The State University

Send all communications concerning Advanced Problems and Solutions to E. P. Starke, Rutgers, The State University, New Brunswick, New Jersey. All manuscripts should be typewritten with double spacing and margins at least one inch wide. Problems containing results believed to be new or extensions of old results are especially sought. Proposers of problems should enclose any solutions or information that will assist the editor. In general, problems in well-known textbooks or results in readily accessible sources should not be proposed for this department.

PROBLEMS FOR SOLUTION

4905. *Proposed by Basil Gordon, Redstone Arsenal, Alabama*

Does the following limit exist:

$$\lim_{x\to 1^-} (1-x)^{1/2} \prod_{n=0}^{\infty} (1+x^{4^n})?$$

4906. *Proposed by P. L. Butzer, Technical University, Aachen, Germany*

If the function $f(x)$ is continuous on an interval (a, b) and, as $h\to 0$, $(1/h^2)\int_0^h [f(x+u)+f(x-u)-2f(x)]\,du\to 0$ for all x in (a, b), then $f(x)$ is a linear function.

4907. *Proposed by P. T. Bateman, University of Illinois*

Suppose $\lambda_1, \cdots, \lambda_r$ are among the elements of a finite abelian group G (written additively) and let H be the subgroup of G generated by $\lambda_1, \cdots, \lambda_r$. Show that there exist positive integers $k_1, \cdots, k_r$ such that every element of H is uniquely expressible in the form

$$n_1\lambda_1 + \cdots + n_r\lambda_r,$$

where n_i is a nonnegative integer less than k_i $(i=1, \cdots, r)$.

4908. *Proposed by John Rainwater, University of Washington, Seattle*

Consider a triangle abc divided into four smaller triangles, a central one def inscribed in abc and three others on the three sides of def. Show that def cannot have the smallest area of the four unless all four are equal with d, e, and f the midpoints of the sides of abc. (The problem seems to have originated with P. Erdös and been transmitted by N. D. Kazarinoff and J. R. Isbell.)

4909. *Proposed by D. J. Newman, Brown University*

Prove that the product of the zeros of $z^n \pm z^{n-1} \pm \cdots \pm z \pm 1$ which lie outside the unit circle is less than $\sqrt{(n+1)}$.

4910. *Proposed by J. S. Frame, Michigan State University*

Find all analytic solutions of the functional equation $f(2z) = 2f(z)f'(z)$.

SOLUTIONS

Derivative of Powers of a Matrix

4823 [1959, 66; 1959, 818]. *Proposed by G. Matthews, St. Dunstan's College, Catford, England*

Let X be a lower-semi-matrix whose elements x_{ij} $(i, j=1, 2, \cdots)$ are independent variables if $j \leqq i$ and zero if $j>i$, and let D be the differential operator matrix defined by $D_{ij}=\partial/\partial x_{ij}$ if $j<i$, $D_{ij}=0$ if $j>i$ and $D_{ii}=\sum_{k=1}^{i} \partial/\partial x_{kk}$. Prove that $D(X^2)=2X$, and hence by induction that $D(X^n)=nX^{n-1}$, where n is any positive integer greater than 2.

II. *Solution by R. D. Gordon, California State Polytechnic College.* There seems to be a gap in the former solution since the relation $DAB=A\cdot DB+DA\cdot B$ is not shown to hold for arbitrary matrices. In fact a counter-example can be cited.

Using the fact that, in the present instance, A and B are powers of the same semi-matrix, the proof can be completed as follows. Let $X^n=(x_{ij}^{(n)})$, then $x_{ij}^{(n)}=0$ if $j>i$. Suppose $DX^n=nX^{n-1}$. This is equivalent to $\sum_{(i\geq k\geq j)} D_{ik}x_{kj}^{(n)}=nx_{ij}^{(n-1)}$. Since

$$X^{n+1}=(x_{ij}^{(n+1)})=X^nX=\left(\sum_{(i\geq l\geq j)} x_{il}^{(n)}x_{lj}\right),$$

it follows that the elements of DX^{n+1} are given by

$$\sum_{(i\geq k\geq j)} D_{ik}x_{kj}^{(n+1)}=\sum_{(i\geq k\geq l\geq j)} D_{ik}x_{kl}^{(n)}x_{lj}=\sum_{(i\geq k\geq l\geq j)} (nx_{il}^{(n-1)}x_{lj}+x_{kl}^{(n)}D_{ik}x_{lj})$$

$$=nx_{ij}^{(n)}+x_{ij}^{(n)}=(n+1)x_{ij}^{(n)}.$$

This is so because the only term in $\sum x_{kl}^{(n)}D_{ik}x_{lj}$ which does not vanish corresponds to $k=i$, $l=j$ and is $x_{ij}^{(n)}(\partial/\partial x_{jj})x_{jj}=x_{ij}^{(n)}$. This gives the desired result $DX^{n+1}=(n+1)X^n$, and the proof is complete since the inductive hypothesis is easily verified for $n=1$.

Finite Ring

4852 [1959, 515]. *Proposed by P. T. Bateman, University of Illinois*

Give an example of a ring with a finite number of elements which cannot be isomorphic to a residue-class ring of the ring of integers of an algebraic number field (of finite degree over the rationals).

Solution by Irving Reiner and Shoji Sato, University of Illinois. Let K be an algebraic number field, I the ring of algebraic integers in K, A an ideal in I. If $A=P_1^{e_1}\cdots P_n^{e_n}$ is the factorization of A into prime ideals P_i in I, then the factor ring I/A is isomorphic with the direct sum S of the rings $I/P_i^{e_i}$, *viz.*,

$$I/A \cong \sum_{i=1}^{n} I/P_i^{e_i} = S.$$

Let $N = \sum_{i=1}^{n} P_i/P_i^{e_i}$. Then N is a subring of S and $N^e = 0$, $N^{e-1} \neq 0$, where $e = \max\{e_1, \cdots, e_n\}$. Also no element of S not in N is nilpotent, so that N is the radical of S. In fact,

$$S/N \cong \sum_{i=1}^{n} I/P_i,$$

where the I/P_i are finite fields. If g_i denotes the number of elements in I/P_i, then the ring $I/P_i^{e_i}$ has $g_i^{e_i}$ elements by the multiplicativity of the ideal-norm.

On the other hand let F be a finite field containing g elements and let $R = F \cdot 1 + Fu_1 + \cdots + Fu_m$ be the commutative algebra over F defined by the multiplication rules

$$1 \cdot u_i = u_i \cdot 1 = u_i \ (i = 1, \cdots, m), \ 1^2 = 1, \ u_i u_j = 0 \ (i, j = 1, \cdots, m).$$

Then the radical $T = Fu_1 + \cdots + Fu_m$ of R satisfies $T^2 = 0$ and $R/T \cong F$. Also R contains g^{m+1} elements.

Now suppose the rings R and I/A of the two preceding paragraphs are isomorphic. Then S/N is isomorphic to a single field F, so that $n = 1$ and $A = P^e$. Since R and I/A must have the same number of elements, $e = m + 1$. Moreover $N^{e-1} \neq 0$, so that $T^m \neq 0$. Thus m must be 0 or 1. Hence, if we take $m > 1$, the ring R provides an example of the type sought.

Dimension of a Banach Space

4853 [1959, 515]. *Proposed by D. J. Newman, Brown University*

The dimension of a Banach space can be defined, *e.g.* in the following two ways: (1) the smallest number (cardinal) of complex numbers necessary to specify a vector; (2) the smallest number of vectors such that linear combinations of them are dense. Are these two definitions equivalent?

Solution by Albert Wilansky, Lehigh University. The definitions are not equivalent. Two dimensional Euclidean space has dimension 2 by the second definition; but dimension 1 by the first, for, since the cardinality of the space is c, it can be put into one-to-one correspondence with the reals, hence any of its points can be specified by one number.

Also solved by Leopold Flatto, and the proposer.

Unitary Matrices Having Preassigned Eigenvalues

4854 [1959, 515]. *Proposed by Ky Fan, Oak Ridge National Laboratory*

Let $\{\alpha_i\}$, $\{\beta_i\}$ $(1 \leqq i \leqq n)$ and γ be given complex numbers, each of absolute value 1. Prove: There exist two unitary matrices A, B of order n with the preassigned eigenvalues $\{\alpha_i\}$, $\{\beta_i\}$, respectively, and such that γ is an eigenvalue

of AB, if and only if, in the complex plane, the convex hull of the n points $\{\alpha_i\}$ and the convex hull of the n points $\{\gamma\bar{\beta}_i\}$ have a point in common.

Solution by the proposer. Let A, B be two unitary matrices of order n with eigenvalues $\{\alpha_i\}$, $\{\beta_i\}$, respectively and such that γ is an eigenvalue of AB. Then γ is also an eigenvalue of BA, so there exists a vector z in the unitary n-space such that $\|z\| = 1$ and $BAz = \gamma z$. Let $Ax_i = \alpha_i x_i$, $By_i = \beta_i y_i$ $(1 \leqq i \leqq n)$, where $\{x_i\}$ are n orthonormal eigenvectors of A, and $\{y_i\}$ are n orthonormal eigenvectors of B. Let $p_i = |(z, x_i)|^2$, $q_i = |(z, y_i)|^2$ $(1 \leqq i \leqq n)$. Then

$$\sum_{i=1}^{n} p_i\alpha_i = \sum_{i=1}^{n} \alpha_i |(z, x_i)|^2 = (Az, z), \qquad \sum_{i=1}^{n} q_i\gamma\bar{\beta}_i = \sum_{i=1}^{n} \gamma\bar{\beta}_i |(z, y_i)|^2 = (\gamma B^*z, z).$$

Since $BAz = \gamma z$, we have $Az = \gamma B^{-1}z = \gamma B^*z$. Hence

$$\sum_{i=1}^{n} p_i\alpha_i = \sum_{i=1}^{n} q_i\gamma\bar{\beta}_i. \tag{1}$$

As $p_i \geqq 0$, $q_i \geqq 0$, $\sum_{i=1}^{n} p_i = \sum_{i=1}^{n} q_i = \|z\|^2 = 1$, equation (1) means that the convex hull of $\{\alpha_i\}$ and the convex hull of $\{\gamma\bar{\beta}_i\}$ have a point in common.

To prove the converse, assume that (1) is satisfied for certain $p_i \geqq 0$, $q_i \geqq 0$ with $\sum_{i=1}^{n} p_i = \sum_{i=1}^{n} q_i = 1$. Let

$$x = \{\sqrt{p_1}, \sqrt{p_2}, \cdots, \sqrt{p_n}\}, \qquad y = \{\alpha_1\sqrt{p_1}, \alpha_2\sqrt{p_2}, \cdots, \alpha_n\sqrt{p_n}\},$$
$$u = \{\sqrt{q_1}, \sqrt{q_2}, \cdots, \sqrt{q_n}\}, \qquad v = \{\gamma\bar{\beta}_1\sqrt{q_1}, \gamma\bar{\beta}_2\sqrt{q_2}, \cdots, \gamma\bar{\beta}_n\sqrt{q_n}\}.$$

Then since $\sum p_i = \sum q_i$ and $|\alpha_i| = |\beta_i| = |\gamma| = 1$, we have $(x, x) = (u, u)$, $(y, y) = (v, v)$. From (1), we have also $(x, y) = (u, v)$. Hence there exists a unitary matrix U of order n such that $Ux = u$, $Uy = v$. Let $A = \text{diag}\,(\alpha_1, \alpha_2, \cdots, \alpha_n)$, $C = \text{diag}\,(\beta_1, \beta_2, \cdots, \beta_n)$ and $B = U^*CU$. Then A, B are unitary matrices with eigenvalues $\{\alpha_i\}$, $\{\beta_i\}$, respectively. We have $\gamma C^*u = v$, which can be written $\gamma C^*Ux = Uy$ or $\gamma x = U^*CUy = By$. But $Ax = y$, so $BAx = By = \gamma x$. Thus γ is an eigenvalue of BA, and therefore also an eigenvalue of AB.

Editorial Note. E. J. Taft and A. C. Mewborn call attention to the fact that the above is a special case of a theorem of H. Wielandt (*Die Einschliessung von Eigenwerten normaler Matrizen*, Mathematische Annalen, v. 121 (1949–1950), p. 234, Satz 1). Wielandt's theorem deals with normal matrices but if the normal matrix is taken as unitary and certain vectors are taken to be unit vectors the adaptation to the present problem imposes little difficulty.

The Growth Function

4856 [1959, 516]. *Proposed by L. A. Rubel, University of Illinois*

The growth function $h(\theta)$ of an entire function $f(z)$ of exponential type is defined by $h(\theta) = \limsup_{r\to\infty} r^{-1} \log |f(re^{i\theta})|$. It is well known that $f(z) \equiv f_1(z)f_2(z)$ need not imply that $h(\theta) \equiv h_1(\theta) + h_2(\theta)$. Suppose, however, that $h_1(\theta) \equiv h_2(\theta)$. Must $h(\theta) \equiv h_1(\theta) + h_2(\theta)$ in this case?

Solution by the proposer. A negative answer is indicated by the following

construction: Choose a large positive number R, and let $A=A_R=a_1, a_2, \cdots$ be the sequence of all positive integers a_n such that for some integer v, $R^{2v}\leqq a_n<R^{2v+1}$, and let $B=B_R=b_1, b_2, \cdots$ be the complementary sequence of positive integers. Put $f_1(z)=\prod(1-z^2/a_n^2)$, $f_2(z)=\prod(1-z^2/b_n^2)$.

Since more sophisticated counterexamples will soon be in print, the proof given here is considerably abbreviated. By symmetry, we need consider only $0\leqq\theta\leqq\frac{1}{2}\pi$, and by continuity of the h-functions, only $0<\theta\leqq\frac{1}{2}\pi$. We use the well-known formula $\log|f_1(re^{i\theta})|=r\int_0^\infty P(t,\theta)D_A(rt)dt$ and the corresponding formula for f_2 in terms of D_B. Here $D_A(x)=x^{-1}\sum_{a_n\leqq x}1$ and $P(t,\theta)=2(1-t^2\cos 2\theta)/(1-2t^2\cos 2\theta+t^4)$. From the fact that $D_B(t)=D_A(Rt)+o(1)$, it follows that $\log|f_2(re^{i\theta})|=R^{-1}\log|f_1(Rre^{i\theta})|+o(r)$ for each $\theta\in(0,\frac{1}{2}\pi]$ as $r\to\infty$, so that $h_1(\theta)\equiv h_2(\theta)$. Clearly, we may also choose R so large that $h_1(\frac{1}{2}\pi)>\frac{1}{2}\pi$. But since $f_1(z)\cdot f_2(z)=\prod(1-z^2/n^2)=\sin\pi z/\pi z$, $h(\theta)\equiv\pi|\sin\theta|$, and thus $h_1(\frac{1}{2}\pi)+h_2(\frac{1}{2}\pi)>\pi=h(\frac{1}{2}\pi)$.

Editorial Note. This proposal was submitted October 1957. J. P. Kahane and the proposer have shown that $h(\theta)=h_1(\theta)+h_2(\theta)$ is false under far more restrictive hypotheses on f_1 and f_2. (See *Sur les produits canoniques de type nul sur l'axe reel*, Comptes Rendus, Acad. Sci. Paris, v. 248, pp. 3102–3103, June 1959. A fuller version will appear in a forthcoming issue of the *Illinois Journal of Mathematics*.) Another example by C. Roumieu is in preparation for the *Annales de l'Ecole Normale Supérieure*.

RECENT PUBLICATIONS

EDITED BY RICHARD V. ANDREE, University of Oklahoma

All books for review should be sent directly to R. V. Andree, Department of Mathematics, University of Oklahoma, Norman, Oklahoma, and not to any of the other editors or officers of the Association.

Mathematics in Physics and Engineering. By J. Irving and N. Mullineux. Academic Press, New York, 1959. xvii+883 pp. $11.50.

This is Volume 6 of the series on Pure and Applied Physics for which H. S. W. Massey is consulting editor. Many topics in advanced calculus for engineers and physicists are found in this remarkable volume: differential equations, Laplace transforms, matrices, classical and quantum mechanics, calculus of variations, complex variable, residues, numerical methods, integral equations. There are 82 pages of appended material, many problems and their solutions. Because of its wide scope it is suitable for classes where shorter texts suitable to individual needs are not available. Teachers would naturally wish to examine the treatment of individual topics for themselves, but this reviewer was very well impressed by the judicious compromise between clear exposition and scholarly depth. Topics seem easy to locate, and the single volume is not too bulky.

ARTHUR BERNHART
University of Oklahoma

Abelian Varieties. (Interscience Tracts in Pure and Applied Mathematics, No. 7.) By Serge Lang. Interscience, New York, 1959, xii+256 pp. $7.25.

It has been over ten years since Weil's definitive *Variétés Abéliennes et Courbes Algébriques* first appeared. During this period a great deal of work has been done on algebraic groups and, in particular, on abelian varieties. Many facets of abelian varieties only mentioned in Weil's treatise have now reached a state of substantial development. This development has been the combined work of the French, Italian, Japanese and American schools. It has therefore become desirable to have a book available which incorporates the results of Weil's treatise with the newer developments in the field of abelian varieties.

Lang has attempted to do just this in the present book and, for the most part, succeeds admirably. Even though—as Lang points out—several special topics are not touched upon, an amazing amount of material is covered. The topics dealt with are the Jacobian, Picard, and Albanese varieties; the 1-adic representations; algebraic systems of abelian varieties. The emphasis is very definitely and very heavily on the "abstract." More examples drawn from "classical" algebraic geometry would help to illuminate and clarify the ideas developed—particularly for the beginner in algebraic geometry. This, however, is a minor flaw in an otherwise excellent book. The reader who is willing to give his time and close attention to this book will be amply rewarded by the insights he will receive into one of the most beautiful and fruitful branches of modern mathematics.

EDWARD H. BATHO
Harvard University

Differential Geometry. By A. V. Pogorelov (translated by Leo F. Boron). Noordhoff, Groningen, 1959. ix+171 pp. $3.90 (paper) or $4.50 (cloth).

This book is an introduction to local theory of curves and surfaces in three-dimensional Euclidean space. The author satisfactorily accomplishes his aim "to present a rigorous discussion of the fundamentals and of the methods of investigation without disturbing well-established tradition in the process." Regular curves and regular surfaces are defined after the manner of a differentiable manifold. The topics are well chosen to cover the essentials in 171 pages which include a bibliography and an index. A large amount of factual material is relegated to exercises and problems at the end of each chapter. Vector notation is used. This book differs from most books at the same level by its concise, rigorous and up-to-date treatment with emphasis on basic concepts. It merits serious consideration as a text or a reference book for beginners in the subject.

However, there are many obvious misprints in the book. In the definition of "order of contact" on page 32, for example, it seems necessary to indicate that n is the greatest integer. Similarly the reader should be aware that some terms like "onto" and "circle" are used in different senses at different places and that the word *respectively* is frequently put in unusual positions.

The chapters in Part One on *Theory of Curves* are: Concept of curve; Concepts for curves which are related to the concept of contact; Fundamental concepts for curves which are related to the concepts of curvature and torsion. Part Two on *Theory of Surfaces* consists of the following chapters: Concept of surface; Fundamental concepts for surfaces which are related to the concept of contact; First quadratic form of a surface and concepts related to it; Second quadratic form of a surface and questions about surface theory related to it; Fundamental equations of the theory of surfaces; Intrinsic geometry of surfaces.

T. K. PAN
The University of Oklahoma

Approximate Methods of Higher Analysis. By L. V. Kantorovich and V. I. Krylov (translated by Curtis D. Benster). Noordhoff, Groningen; Interscience, New York, 1958. 681 pp. $17.00.

This is the English translation of the third Russian edition (1950) of the book "Priblizhennye metody vysshego analiza" by the two famous authors. On the jacket, the book is praised as a "unique text and reference book of pure and applied mathematics." It can be considered as an almost unique text because of the wealth of information contained therein (for a table of content see the Mathematical Reviews, vol. 13, 1952, p. 77) and the great number of carefully worked out examples, many of which are not trivial. It certainly has to be considered as a "unique" reference book due to the complete absence of a subject index. This fact makes its use as a reference book very awkward indeed, especially as the section headings in the contents are often ambiguous and prevent an accurate guess as to their content. The translation is precise and clear, but rather stilted due to the translators obvious aim to stick to the Russian text as closely as possible. If the translator had supplied appropriate references to Anglo-American literature, the usefulness of this work would have been increased many fold. The printing is clear except for a surprisingly large number of space bar blurs. The formulas and tables are well arranged.

HANS SAGAN
University of Idaho

Combinatorial Topology, Vol. 1. By P. S. Aleksandrov. Graylock Press, Rochester, New York, 1956. xvi+225 pp. $4.95.

Combinatorial Topology, Vol. 2, *The Betti Groups*. By P. S. Aleksandrov. Graylock Press, Rochester, New York, 1957. xi+244 pp. $6.50.

These two volumes are a translation of Parts I, II and III of Aleksandrov's *Kombinatornaya Topologiya*. The translation into English has been made with a commendable lack of awkwardness, resulting in a masterful treatise, moderately paced. The book is designed to meet the needs of the student with maturity who has only modest familiarity with general topology and the concepts of algebra. Except for Chapter I, which is of an introductory nature, the arguments

are made in considerable detail. The material is beautifully organized, and there are numerous remarks and examples by way of clarification. There are very few exercises, but in all other respects the book constitutes an excellent textbook on classical homology theory.

Chapter II is devoted to proving the Jordan curve theorem. Chapter III covers the topological classification of closed two-dimensional surfaces. Euler's theorem (that two triangulations of a surface have the same Euler characteristic) is employed without proof to show that the connectivity of a surface is invariant under homeomorphism. (A proof is supplied later in Chapter X.) Chapter IV is a detailed study of complexes and Chapter V treats Sperner's lemma and its corollaries, principally the Pflastersatz, the invariance of interior points of subsets of R^n, and the fixed-point theorem for n-cells. Chapter VI is an introduction to dimension theory and treats only the covering dimension for compacta. These six chapters, with an appendix on n-dimensional geometry, complete Volume I.

Volume 2 (Part III of the original work) is devoted to the Betti groups. The notions of chain and boundary operator, and the resultant algebraic apparatus are developed in Chapter VII. The elementary theory of homology groups is developed in Chapter VIII and in Chapter IX the cohomology groups appear and are used to obtain relations among homology and cohomology groups with different coefficient domains. The invariance of the Betti groups is proved in Chapter X. In Chapters XI and XII the homology theory is extended from polyhedra to compacta. An appendix is devoted to basic information on Abelian groups.

Volume 3 has not yet appeared in English translation. Since Volume 2 is introductory to the remaining chapters and, in a sense, is motivated by what is to follow, it is to be hoped that the remainder of the book will be published in English at an early date. Among other topics, these chapters contain the duality theories of Poincaré and Alexander-Pontryagin, and the Lefschetz fixed-point formula.

L. E. WARD, JR.
University of Oregon

Industrial Mathematics. By Matthias Thies. Prentice Hall, Englewood Cliffs, N. J., 1959, 342 pp. $9.00.

A more appropriate title for this book would be "Industrial Arithmetic", for there is no real mathematics in it. The basic operations of arithmetic and the computational aspects of trigonometry are treated in very great detail. The author attains this simplicity in his presentation by paying the price of careless statements.

This book is definitely not suitable as either a high school or college text. This reviewer was also appalled at the high price of the book, considering the caliber of the contents.

LEONARD E. FULLER
Kansas State University

Elementary Decision Theory. By Herman Chernoff and L. E. Moses. Wiley, New York, 1959. xv+364 pp. $7.50.

The day of determinism is past, even in the field of engineering, and it is rather amusing to remark that the determinist has inevitably brought it on himself. By constructing more and more realistic models of the physical universe, of greater and greater complexity, he has slowly forced himself into a position where the only escape from stalemate lies in the introduction of new rules into the game. The result is that we must coat ignorance of true cause and effect and inability to handle the resultant equations, even if we know them, by the soothing, albeit sophisticated, salve of probability theory.

The modern engineer who is interested in feedback control processes, in automation, in processes involving adaptive control, and in any number of significant uses of digital and analogue computers, must acquaint himself with the mathematical theory of decision processes. He must understand the difficulties and advantages of uncertainty, and be familiar with the various tricks that are used to construct precise mathematical models of imprecise situations.

Unfortunately, many of the standard works in the field of decision-making start from a high mathematical level and continue upward. The result is that they are not suited to the needs of the beginner.

The book by Chernoff and Moses is a book designed specifically for the scientist interested in decision processes, but possessing only a rudimentary mathematical background. The authors have succeeded admirably in what they have set out to do. Carefully and slowly, they illustrate how mathematical models of decision-making under uncertainty are formulated. They discuss the choice of a criterion function, the choice of possible policies, the description by means of state variables, and so on.

All of this is done in a very pleasant, readable style, with numerous examples, and much important discussion. Most important, they are honest with the reader in constantly pointing out pitfalls and limitations. Their presentation illustrates very clearly that fundamental ideas can be transmitted without the pseudo-abstraction that befogs the Bourbaki and their devotees.

In the second half of the book, they give an introduction to classical statistics. Throughout, their aim is to follow the guiding idea of Wald, who conceived of statistics as decision-making under uncertainty. In their presentation they have followed the teachings of the late M. A. Girshick.

The book is recommended to mathematicians, physicists, engineers, and so forth, who wish to obtain a relatively painless introduction to this modern field, either for their own studies, or merely to keep abreast of current intellectual activity.

RICHARD BELLMAN
The RAND Corporation

La Localisation des Valeurs Caractéristiques des Matrices et ses Applications. By M. Parodi. Gauthier-Villars, Paris, 1959. x+172 pp. $7.76.

This book is an almost-complete survey of results on the determination of bounds for the eigenvalues of a square matrix which have been found by such workers as A. Ostrowski, A. Brauer, K. Fan, O. Taussky and the author, and which have as their prototype the well-known "Geršgorin circles" theorem. Some of the topics treated are: sufficient conditions for a matrix to be nonsingular, bounds (in the complex plane) for the eigenvalues of a matrix, sufficient conditions for the stability of an analogue computer, the location of the zeros of polynomials, the determination of the zeros of determinants with polynomial elements.

Perhaps the most interesting section is that concerned with polynomials, in which much unfamiliar and inaccessible material is given on bounds for the zeros of a polynomial in terms of its coefficients.

Anyone concerned with the numerical determination of eigenvalues will find this a useful and practical reference work.

MORRIS NEWMAN
National Bureau of Standards

A Philosopher Looks at Science. By John G. Kemeny. Van Nostrand, Princeton, N. J., 1959. 273 pp. $4.95.

Mathematical journals seldom find time to review books on philosophy or even the philosophy of science, but the author of this one, while having every right to call himself a philosopher, happens to be a mathematician known for his research in logic and contributions to undergraduate education. In the first part of the book, "What Science Presupposes," Kemeny speaks primarily as a mathematician with a philosophical bent, or perhaps one should say as a philosopher thoroughly imbued with mathematics and making full use of it to deflate pseudo-problems and to formulate ideas simply. In Part two, "Science," and in the third and final part, "Problems Raised by Science," the philosopher in Kemeny comes more to the fore, though even here there are constant reminders of his mathematical competence and point of view.

The book is essentially a popularization, directed to the interested layman or the undergraduate student. However, important ideas can be presented in a book at this level, and Kemeny writes in his preface that he hopes his book will be of some interest even to the expert. Possibly Kemeny could be accused of oversimplification and of sidestepping really difficult philosophical problems by redefinition. But this reviewer finds the discussion convincing and satisfying—it is difficult to be critical of an author whose ideas one finds congenial.

The mathematician should find the book enjoyable, since it presents a point of view that is consistent with mathematical knowledge and inclinations.

KENNETH O. MAY
Carleton College

General Homogeneous Coordinates in Space of Three Dimensions. (paper edition.) By E. A. Maxwell. New York: Cambridge University Press, 1959. xiv+165 pp. $2.75.

This is a sequel to the author's earlier work on the methods of plane projective geometry based on the use of general homogeneous coordinates (Cambridge University Press, 1946). A knowledge of the material in the earlier volume is assumed. The principal emphasis is on projective geometry of ordinary space. One chapter portrays applications to euclidean geometry.

It is curious that the author replaces the usual statement concerning linear dependence of elements by the remark that the elements are in *syzygy*. It is regrettable that use is not made of a notation which can be extended easily to the case of n-dimensions. The eighth and last chapter introduces matrices, and contains a clever and novel treatment of line coordinates. The belated mention of matrices deprives the reader of an opportunity to gain practice in the use of this tool. Many standard exercises appear under the heading of Theorem-examples. More difficult exercises are found at the end of each chapter. Following a development of quadric surfaces, there is a treatment of line geometry including the linear and tetrahedral complexes. The twisted cubic and systems of quadrics are studied.

There is much of interest in this well-written account of a selection of topics from classical algebraic geometry. It should be a welcomed addition to the literature because of its uniqueness of approach and its value to one who aspires to read more advanced treatises on geometry.

C. E. Springer
University of Oklahoma

Introduction to Analysis. By N. B. Haaser, J. P. LaSalle, J. A. Sullivan. Ginn, Boston, 1959. 688 pp. $8.50.

This is a skillfully written rigorous presentation of introductory analysis, which should prove highly satisfactory for a student who has had thorough preparation in high school mathematics, and who is guided in this volume by a competent teacher. Even with the omission of about fifty pages which the preface indicates as optional, there is more here than can be covered in two five-hour semesters.

The language of sets is introduced at the beginning and is used throughout. Then the axioms for the real numbers (except the least-upper-bound axiom which is delayed appropriately until page 381) are stated, and the following treatment of algebra, geometry, trigonometry, and calculus is based upon this set of axioms. Vectors as ordered pairs of real numbers are employed extensively in the development of analytic geometry and trigonometry. A function is defined as a "set of ordered pairs of elements such that no two distinct pairs have the same first element." A transformation of the Euclidean plane R^2 is a function T with domain R^2 and range R^2, so that transformations constitute a special

class of functions. Following an intuitive discussion of arc length, the real number line R is mapped on the unit circle. To a number u on R there corresponds a point on the circle. A function $F(u)$ is thus defined, the ordered pair for which are cos u and sin u. The law of sines and the law of cosines are developed by vector methods. Mathematical induction is treated rigorously and a careful foundation for the calculus is laid. Even the Heine-Borel theorem, and uniform continuity appear, although these are in sections described as optional.

It is curious to find a discussion of the solution of a quadratic equation following chapters involving real variable theory, problems on maxima and minima, and the graphing of functions more complicated than the quadratic function. Of course, this is to open the way for complex numbers, which are defined as ordered pairs of real numbers with certain operations of addition and multiplication. Complex numbers are shown to be an instance of a field. The logarithm, exponential, and inverse functions are introduced near the end of the volume. Such topics as infinite series and partial differentiation are not treated. These will presumably appear in the projected Volume II. The Taylor theorem is introduced and is used in a discussion of numerical integration.

The authors and publishers are to be commended for producing this excellent volume which is in the spirit of contemporary mathematics. All teachers of college mathematics should be familiar with its contents.

C. E. SPRINGER
The University of Oklahoma

Introductory Calculus. By Donald E. Richmond. Addison-Wesley, Reading, Mass., 1959. xv+207 pp. $6.75.

Richmond's *Introductory Calculus* is refreshingly different, and it merits special attention for its faithfulness to the spirit of modern mathematics. Recognizing that "Mathematics is concerned with logical consequences of different sets of assumptions," the author states assumptions and definitions with precision, and he leaves no doubt about the logical status of statements which are made. Theorems are proved with unusual care, but there is no quixotic tendency to make a fetish of rigor. The stage for an important idea or proof generally is first set by an intuitive approach which is fortified by the sound conviction that "it is important to lay bare the essential idea as clearly as possible."

This text, consisting of the first five chapters of Richmond's *Calculus* (appearing also in 1959), is intended to serve as the basis for a semester course, and as such it should meet with favor. An instructor who does not adopt it as the textbook for the course may yet wish to use it as supplementary material, for it affords delightfully different approaches and unconventional proofs. The power rule for the differentiation of positive integral powers of functions, for example, is derived through mathematical induction from the product rule; and the derivative of the logarithmic function is obtained as the inverse of an exponential function. More strikingly novel are the definitions of the sine and

cosine functions as the imaginary and real parts of a number representing a point on a unit circle in the complex plane, from which the derivatives and other properties are obtained.

Richmond's *Introductory Calculus* resembles, in size, scope, and purpose, the first six chapters of his *Fundamentals of the Calculus* (1950), but it is clearly a new and more sophisticated volume. There is increased confidence in novelty of treatment, and there is greatly heightened emphasis upon depth of understanding.

A few peccadillos may catch the eyes of an attentive reader—the ubiquitous use of "may" for "can," an inadequacy in the definition of point of inflection (p. 68), and the reversal of the inequality signs in Archimedes' bounds for π (p. 159). These will in no way jeopardize the final judgment on the book as a sound work which exhibits major expository and mathematical virtues.

CARL B. BOYER
Brooklyn College

Contributions to the Theory of Games, Vol. IV. Edited by A. W. Tucker and R. D. Luce. Annals of Mathematics Study No. 40, Princeton University Press, 1959. vii+453 pp. $6.00.

This is the fourth (and last) volume of the series *Contributions to the Theory of Games*. It carries a dedication to the late John von Neumann and is devoted entirely to an area in which he took a special interest, the theory of non-strictly competitive games. These games, which are roughly all games except zero-sum two-person games, bear the greatest promise for fruitful application to the social sciences. At the same time, this is the branch of game theory that is least developed mathematically and which is replete with problems unformulated or unsolved. All of the current directions of research are well represented in this collection of nineteen papers by sixteen authors.

A detailed review of the contents is rendered superfluous by the editors' introduction, which provides a synopsis of each paper and arranges them in a natural order. The contributions fall into two broad classes: those which investigate the von Neumann-Morgenstern concept of a solution and those which attempt to modify or improve it. Unfortunately, the question of the existence of a von Neumann-Morgenstern solution for every finite game is still open.

With its wide coverage of research, this is an indispensable book for anyone who wishes to keep abreast of progress in the theory of games. It also includes two bonus features, which make it even more valuable: a translation of von Neumann's classic 1928 paper, which initiated the modern theory of games, and a bibliography on game theory with more than 1,000 entries.

HAROLD W. KUHN
Princeton University

NEWS AND NOTICES

EDITED BY LLOYD J. MONTZINGO, JR., University of Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to L. J. Montzingo, Jr., University of Buffalo, Buffalo **14**, *New York. Items must be submitted at least two months before publication can take place.*

SUMMER SESSIONS

University of California, Los Angeles, June 20 to July 29: Professor Horn, functions of a complex variable; Visiting Professor Grünbaum, linear programming.

University of Illinois, June 20 to August 13: Professor Day, combinatorial topology; Professor Ketchum, measure and integration; Professor Ribenboim, theory of rings; Professor Brahana, linear transformations and matrices; Professor Schubert, introduction to higher algebra; Professor Reiner, number, length, area. In addition, the following courses will be given: fundamental concepts of geometry; advanced algebra; introduction to higher geometry; complex variables and applications; introduction to higher analysis; advanced statistics; mathematical methods in engineering and science.

State University of Iowa, June 13 to August 10: Professor Dye, introduction to analysis, introduction to algebraic topology; Professor Hogg, topics in mathematical statistics, statistical hypotheses, seminar in mathematical statistics; Dr. Jakobsen, differential equations, matrices and determinants; Professor Muhly, elementary theory of numbers, topics in commutative algebra, seminar in algebra; Professor Price, construction of technical aids in the teaching of mathematics, supervision of mathematics; Professor Reid, integral equations, seminar in analysis.

University of Michigan, June 20 to August 13: Dr. Chu, vector analysis; Professor Craig, statistical analysis, mathematical theory of probability; Professor Dushnik, operational mathematics, topics in modern mathematics for teachers; Professor Dwyer, calculus of finite difference; Professor Griffin, advanced calculus; Dr. Hedstrom, differential equations; Dr. Hicks, fourier series and applications; Professor D. Jones, mathematical theory of probability, theory of statistics; Professor P. Jones, history of mathematics; Dr. Kincaid, introduction to functions of a complex variable with applications; Dr. Kister, set-theoretic topology; Professor Livingstone, theory of equations and determinants, topics in geometry for teachers; Professor McLaughlin, introduction to matrices, introduction to theory of numbers; Mr. Morrill, mathematics of life insurance; Dr. Mrowka, introduction to differential equations, introduction to the foundations of mathematics; Professor Murdoch, introduction to matrices, algebra; Dr. Ramanujan, advanced mathematics for engineers; Professor Reade, introduction to functions of a complex variable with applications, differential geometry; Dr. Rosen, theory of dimensions; Dr. Schaetz, operational mathematics, intermediate course in differential equations; Dr. Shimrat, advanced mathematics for engineers; Professor Ullman, fourier series and applications, complex analysis; no instructor, introduction to differential equations; no instructor, differential equations.

PERSONAL ITEMS

Dr. R. W. Bass, RIAS, Baltimore, Maryland, has been appointed Chief Scientist at Aeronca Manufacturing Corporation, Aerospace Division, Baltimore, Maryland.

Mr. L. W. Chinnery, Grinnell College, has accepted a position as Computer Programmer Trainee with the System Development Corporation, Santa Monica, California.

Mr. R. H. Christ, RCA, Camden, New Jersey, has been appointed Data Reduction Mathematician with the RCA Missile Test Project, Patrick Air Force Base, Florida.

Mr. E. A. Covey, Abilene Christian College, has been appointed Mathematician with the U. S. Civil Service, White Sands Missile Range, New Mexico.

Mr. Charles Drescher, Ramo-Wooldridge Corporation, Los Angeles, California, has accepted a position as Electrical Systems Engineer with the System Development Corporation, Lodi, New Jersey.

Mr. W. G. Fair, University of Missouri, has accepted a position as Junior Engineer with the Bendix Aviation Corporation, Kansas City, Missouri.

Mr. S. I. Gass, Operations Research Branch, Council for Economic and Industrial Research, Arlington, Virginia, has accepted a position as Senior Mathematician with the Federal Systems Division of the I.B.M. Corporation, Washington, D. C.

Mr. R. Q. Jennett, The Martin Company, Florida, has accepted a position as Senior Aerophysics Engineer for Convair, Fort Worth, Texas.

Dr. A. T. Lauria, Purdue University, has accepted a position as Mathematician with the Research Planning Group of the Linde Company, Tonawanda, New York.

Mr. C. D. Robbins, Douglas Aircraft, Santa Monica, California, has accepted a position as Research Engineer with the Temco Aircraft Company Experimental Laboratories, Dallas, Texas.

Professor J. W. Tukey, Princeton University, has been appointed to membership on President Eisenhower's Science Advisory Committee for the four-year term beginning January 1, 1960.

Mr. W. S. Wenger, University of Kansas, has accepted a position as Mathematician in the Ballistic Research Laboratory, Army Proving Ground, Aberdeen, Maryland.

Dr. Albert Wolinsky, General Precision Laboratory Inc., Pleasantville, New York, has been promoted to Senior Staff Member.

THE MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

NEW MEMBERS

Professor H. L. Alder, Secretary, announces that the following 345 persons have been elected to membership by the Board of Governors on applications duly certified.

CESAR A. ABUAUAD, Licenciado (Chile) Prof., University of Chile

ALFRED AEPPLI, Ph.D. (Swiss Federal Inst. Tech.) Asst. Prof., Cornell University

JOSIAH P. ALFORD, M.A. (George Washington) Grad. Asst., Pennsylvania State University

FARID R. AHMED, Ph.D. (Leeds U.) Asso. Research Officer, National Research Council, Canada

ARNOLD O. ALLEN, M.A. (California, Los Angeles) Fellow, University of California, Los Angeles

CLYDE E. ALLEN, JR., B.S. (Gustavus Adolphus Coll.) Mathematician, Remington Rand Univac

BINYAMIN A. AMIRA, D.Sc. (Geneva) Asso. Prof., Hebrew University, Israel

EDWARD S. K. ANSAH, A.B. (Wilkes Coll.) Grad. Asst., Michigan State University

MRS. LUCEIL K. ARNETT, M.A. (West Virginia) Instr., American University

CLAYTON V. AUCOIN, Ph.D. (Auburn University) Asst. Prof., Southwestern Louisiana Institute

JASON M. AUSTIN, Colonel, U. S. Army, Retired

JOHN D. AYDLOTT, A.B. (Wayne S.U.) Instr., Highland Park Junior College

LOTFALI BADII, Student, San Jose State College

IRVING F. BARDITCH, B.S.E.E. (Johns Hopkins) Senior Engr., Westinghouse Electric Corporation

LEE W. BARIC, B.S. (Dickinson Coll.) Instr., Lafayette College

LARRY BARNES, Student, Classen High School, Oklahoma City

BRUCE R. BARRETT, Student, University of Kansas

JAMES E. BARRY, B.S. (Carnegie Inst. Tech.) Engr., E. I. duPont de Nemours and Company

ANTHONY R. BATTISTA, B.S. (St. Francis Coll.) Jr. Mathematician, Teleregister Corporation

GEORGE F. BEATTIE, M.A. (Claremont Coll.) Instr., San Bernardino Valley College

THEODORE S. BECK, Student, San Diego State College; Sr. Research Engr., Ryan Aeronautical Co.

DALE R. BEDGOOD, M.A. (Arkansas) Grad. Asst., University of Oklahoma

LELAND K. BELDEN, B.A. (Iowa S.C.) Iowa City

CLARENCE W. BENNETT, M.S. (Massachusetts) State Supervisor of Mathematics, Maine

GORDON M. BENNETT, M.A. (Columbia T.C.) Asst. Prof., Ohio Northern University

MORRIS M. BENNETT, Student, Long Beach State College

JAMES A. BJUSTROM, M.A. (Columbia) Mathematician, Remington Rand Univac

LESTER M. BLUM, B.E.E. (C.C.N.Y.) Member, Technical Staff, Hughes Aircraft Company

LESLIE E. BLUMENSON, M.S. (N.Y.U.) Staff Mathematician, Columbia University

RAJ C. BOSE, D.Litt. (Calcutta) Prof., Statistics, University of North Carolina

ROBERT N. BOYD, B.A.S. (Toronto) Supervising Engr., DuPont of Canada

RICHARD B. BRIAN, M.A. (Maryland) Asso. Mathematician, Applied Physics Lab., Johns Hopkins University

RAYMOND F. BRINKER, Student, North Central College

BROTHER ALBAN ALBRIGHT, Ph.D. (Pennsylvania) Asst. Prof., La-Salle College

BROTHER PATRICK MATTHEW, B.A.

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JOHN ALAN BROWN, M.D. (Minnesota) Instr., Montana State College

RICHARD H. BROWN, Ph.D. (Columbia) Asst. Prof., Washington College

STEFAN A. BURR, Student, University of California, Berkeley

FAY B. BURRAS, Student, Lebanon Valley College

SYLVIA BURTON, B.A. (St. Joseph's Coll.) Grad. Student, New York University

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BENJAMIN P. CARTER, Student, Stanford University

CHARLES CASTONGUAY, Student, University of Ottawa

ERNEST F. CHACE, Student, Oklahoma State University

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WILLIAM J. CONGDON, Ph.D. (Cornell) Manager of Estimating, Allied Chemical Corporation

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GARY D. CROWN, Student, University of Wichita

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WILLIAM W. HAMILTON, M.A. (Texas) President, Royal Crown Bottling Company

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THE JANUARY MEETING OF THE NORTHERN CALIFORNIA SECTION

The twenty-second annual meeting of the Northern California Section of the Mathematical Association of America was held at the University of California, Berkeley, January 16, 1960. Professor G. C. Preston, Chairman of the Section, presided. There were 153 persons in attendance, including 131 members of the Association.

At the business meeting a report of the Committee to Study the Activities was presented; an amendment to the Section by-laws was adopted; and, under the new by-laws, the following officers were elected: Chairman and Program Chairman elect, Professor S. P. Hughart, Sacramento State College; Vice-Chairman and Chairman elect, Professor David Blakeslee, San Francisco State College; Program Chairman, Professor G. C. Preston, San Jose State College; Secretary-Treasurer, Professor Roy Dubisch, Fresno State College. The term of the Secretary-Treasurer will be for three years and the others for one year. In the future, under the new by-laws, the only election will be that for Vice-Chairman except in the years when the term of the Secretary-Treasurer expires.

By invitation of the section, two addresses were given. Professor A. N. Milgram, University of Minnesota and University of California, Berkeley, spoke on *The Heat Equation* and traced the influence of this equation on the development of modern points of view on analysis. Professor G. B. Price, University of Kansas and California Institute of Technology, spoke on *Generalized Area and Certain Formulas in Calculus*. An abstract of this address follows:

A ribbon, a special type of surface determined by two space curves, has an invariant called generalized area. Under suitable hypotheses, generalized area can be expressed as a definite integral. Conditions are determined under which generalized area reduces to ordinary area. It is shown that the integral formula for generalized area reduces in special cases to the integral formulas for the following: area of a surface of a cylinder, a cone, and a frustum of a cone; area under a curve in the plane; area in polar coordinates; and the area enclosed by a curve in the plane.

The following papers were presented:

1. *The centroid in absolute geometry*, by Professor C. M. Fulton, University of California, Davis.

The following theorem is proved: The lines joining the midpoints of opposite edges of a tetrahedron intersect in the centroid. For the proof only Hilbert's axioms of incidence, order, and

congruence are used. No assumption about continuity and parallelism is required. The main tool is a known theorem of absolute geometry, namely: The line joining the midpoints of two sides of a triangle is orthogonal to the perpendicular bisector of the third side.

2. *Remarks on the S.M.S.G. Program in Northern California*, by Professor Bernard Friedman, University of California, Berkeley, and Mr. Robert Starkey, Cubberley Senior High School, Palo Alto.

3. *An integral inequality derived from convexity*, by Dr. Albert Novikoff, Stanford Research Institute, Menlo Park.

A well-known inequality (due to Young) between two functions, each the inverse of the other, has been generalized recently by Fenchel. The resulting inequality between functions of n variables is geometrically interpretable and seems more natural than earlier attempts at generalization (cf. Hardy, Littlewood, and Polya's *Inequalities*). An interesting by-product is the ability to find the potential function for certain conservative fields without the necessity of integration.

4. *Satellite orbits*, by Dr. J. L. Brenner, Stanford Research Institute.

Finite (closed-form, approximate) equations for the motion of a near satellite of the earth have been derived. The analysis parallels and refines that of King-Hele, Proc. Roy. Soc. (A) 247, 49–72 (1958). The equations have been checked by numerical integration of the differential equations. In view of the form of the equations, the "pear-shaped" theory of the earth's gravitational field does not seem to be established. The motion of satellite 1958 Beta Two can be explained on the basis of an earth axially symmetric, and with northern and southern hemispheres congruent.

ROY DUBISCH, *Secretary*

CALENDAR OF FUTURE MEETINGS

Forty-first Summer Meeting, Michigan State University, East Lansing, Michigan, August 29–September 1, 1960.

Forty-fourth Annual Meeting, Willard Hotel, Washington, D. C., January 25–27, 1961.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Associate Secretary.

ALLEGHENY MOUNTAIN
ILLINOIS
INDIANA
IOWA
KANSAS
KENTUCKY
LOUISIANA-MISSISSIPPI, Buena Vista Hotel, Biloxi, Mississippi, February 17–18, 1961.
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA
METROPOLITAN NEW YORK
MICHIGAN
MINNESOTA
MISSOURI
NEBRASKA
NEW JERSEY
NORTHEASTERN, Wesleyan University, Middletown, Connecticut, November 26, 1960.
NORTHERN CALIFORNIA, San Jose State College, January 14, 1961.
OHIO
OKLAHOMA
PACIFIC NORTHWEST, State University of Montana, Missoula, June 17, 1960.
PHILADELPHIA, Swarthmore College, Swarthmore, Pennsylvania, November 26, 1960.
ROCKY MOUNTAIN
SOUTHEASTERN
SOUTHERN CALIFORNIA
SOUTHWESTERN
TEXAS
UPPER NEW YORK STATE
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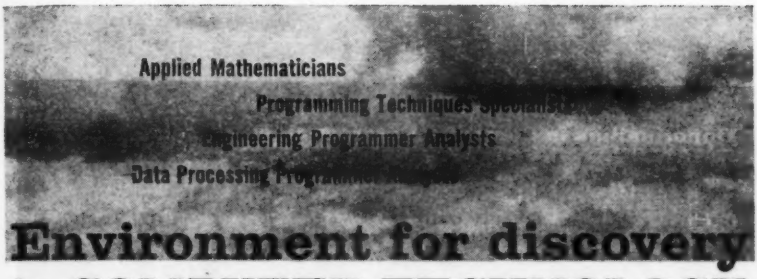

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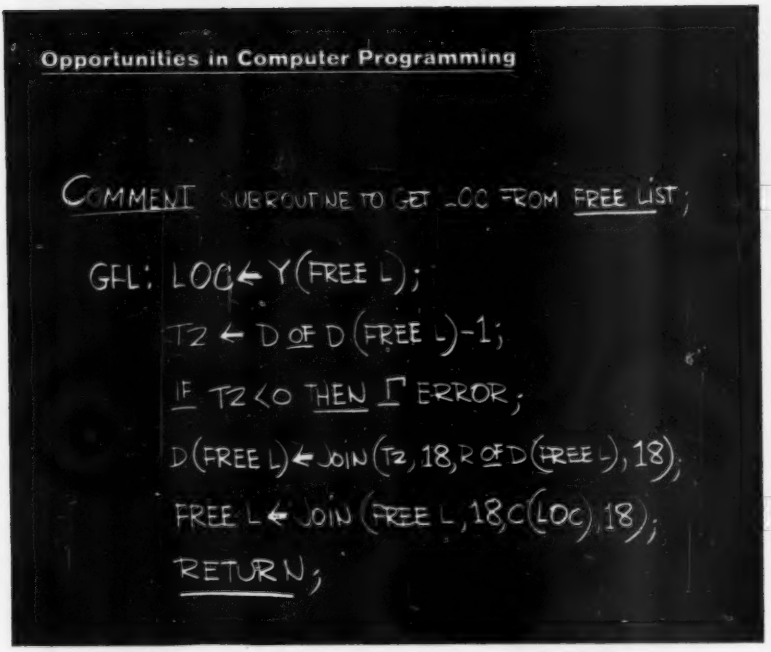
COMMENT SUBROUTINE TO GET LOC FROM FREE LIST;
GFL: LOC ← Y(FREE L);
T2 ← D OF D(FREE L)-1;
IF T2<0 THEN ⌐ERROR;
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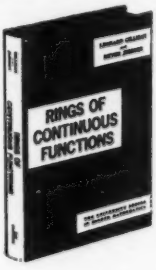

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